LATTICES IN *Sol*

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Definition 1. Let (G, dg) be a Lie group with a left invariant Riemannian metric. A subgroup $\Gamma \leq G$ is said to be *discrete* if the induced subset topology on Γ is discrete. Since dg is left invariant there is an induced metric on the quotient space G/Γ, where Γ acts on G by left translation. We say that Γ is a *lattice* in G if Γ is a discrete subgroup and $\mathrm{Vol}(G/\Gamma)$ is finite.

Note in particular that if $\Gamma \leq G$ is a discrete subgroup with G/Γ compact then Γ is a lattice in G.

Definition 2. By *Sol* we mean the Lie group $\mathbb{R}^2 \rtimes \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ as $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$, so as multiplication is given by $(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + e^{t_1} x_2, y_1 + e^{-t_1} y_2, t_1 + t_2)$, together with the left invariant Riemannian matric $ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2$.

The metric on *Sol* is constructed from a collection of trivializing left invariant vector fields as follows. Consider the three curves $\mathbb{R} \to Sol$ given by $\gamma_1 : s \mapsto (s, 0, 0)$, $\gamma_2 : s \mapsto (0, s, 0)$ and $\gamma_3 : s \mapsto (0, 0, s)$. These have tangent vectors $\frac{\partial \gamma_1}{\partial s} = \frac{\partial}{\partial x}$, $\frac{\partial \gamma_2}{\partial s} = \frac{\partial}{\partial y}$ and $\frac{\partial \gamma_3}{\partial s} = \frac{\partial}{\partial t}$ at $(0, 0, 0)$ respectively, and these span the tangent space at that point. The left action of the group on these vectors gives a collection of three left invariant vector fields X_1, X_2 and X_3 which form a basis for the tangent space at each point. $(x, y, t)\gamma_1 : s \mapsto (x + e^t s, y, t)$ so $X_1(x, y, t) = \frac{\partial}{\partial s}\{(x, y, t)\gamma_1\}|_{s=0} = e^t \frac{\partial}{\partial x}$. Similarly $(x, y, t)\gamma_2 : s \mapsto (x, y + e^{-t}s, t)$ so $X_2(x, y, t) = \frac{\partial}{\partial s}\{(x, y, t)\gamma_2\}|_{s=0} = -e^{-t}\frac{\partial}{\partial y}$ and $(x, y, t)\gamma_3 : s \mapsto (x, y, t + s)$ so $X_3(x, y, t) = \frac{\partial}{\partial s}\{(x, y, t)\gamma_3\}|_{s=0} = \frac{\partial}{\partial t}$. We construct the metric to be orthogonal at every point with respect to these vector fields. Thus $(\frac{\partial}{\partial x}|_{(x,y,t)}, \frac{\partial}{\partial x}|_{(x,y,t)}) = (e^{-t}X_1(x, y, t), e^{-t}X_1(x, y, t)) = e^{-2t}$, $(\frac{\partial}{\partial y}|_{(x,y,t)}, \frac{\partial}{\partial y}|_{(x,y,t)}) = (-e^t X_2(x, y, t), -e^t X_2(x, y, t)) = e^{2t}$ and $(\frac{\partial}{\partial t}|_{(x,y,t)}, \frac{\partial}{\partial t}|_{(x,y,t)}) = (X_3(x, y, t), X_3(x, y, t)) = 1$ and so we obtain the metric given above.

Proposition 3. *Let $A \in SL_2(\mathbb{Z})$. Suppose that A is conjugate in $GL_2(\mathbb{R})$ to a matrix of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for some $\lambda \neq 1$. Then there is a quasi-isometric embedding $\mathbb{Z}^2 \rtimes_A \mathbb{Z} \hookrightarrow Sol$ and the image is a lattice. In particular if A and B are both matrices of the above form then $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is quasi-isometric to $\mathbb{Z}^2 \rtimes_B \mathbb{Z}$.*

Note that by $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ we mean the semidirect product where $t \in \mathbb{Z}$ acts on $\mathbb{Z}^2$ as A^t so as multiplication is given by $(x_1, y_1, t_1)(x_2, y_2, t_2) = ((x_1, y_1) + A(x_2, y_2), t_1 + t_2)$.

Proof. By assumption there exists $P \in GL_2(\mathbb{R})$ such that $PAP^{-1} = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and $s \in \mathbb{R} \setminus \{0\}$ such that $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} e^s & 0 \\ 0 & e^{-s} \end{smallmatrix}\right)$. Define the embedding by $(x, y, t) \mapsto (P(x, y), st)$ and note that since $s \neq 0$ and P is nonsingular this is an injection. The following calculation demonstrates that this gives a homomorphism: $(x_1, y_1, t_1)(x_2, y_2, t_2) = ((x_1, y_1) + A^{t_1}(x_2, y_2), t_1 + t_2) \mapsto (P(x_1, y_1) + PA^{t_1}(x_2, y_2), s(t_1 + t_2)) = (P(x_1, y_1) + \left(\begin{smallmatrix} e^s & 0 \\ 0 & e^{-s} \end{smallmatrix}\right)^{t_1} P(x_2, y_2), st_1 + st_2) = P(x_1, y_1) + \begin{pmatrix} e^{st_1} & 0 \\ 0 & e^{-st_1} \end{pmatrix} P(x_2, y_2), st_1 + st_2) = (P(x_1, y_1), st_1)(P(x_2, y_2), st_2)$. The quotient of *Sol* by $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is a $\mathbb{T}^2$ bundle over S^1 so is compact. Thus $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is indeed a lattice in *Sol*.

We now show that the action of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ on *Sol* is proper. Thus let $p = (X, Y, T) \in Sol$ and let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{Z}^2 \rtimes_A \mathbb{Z}) \setminus \{1\}$. Then $\gamma p = (P(\gamma_1, \gamma_2) + (e^{\gamma_3} X, e^{-\gamma_3} Y), s\gamma_3 + T)$. If $\gamma_3 \neq 0$ then $d(p, \gamma p) \geq |s| \geq 0$. If $\gamma_3 = 0$ then $\gamma p = (P(\gamma_1, \gamma_2) + (X, Y), T)$ and both p and γp lie in the same horizontal plane $t = T$ on which the metric restricts to $ds^2 = e^{-2T} dx^2 + e^{2T} dy^2 + dt^2$. In this case let $\mu = \min\{e^{-2T}, e^{2T}\} > 0$ and let $K = \inf_{\|(x,y)\|_2 = 1} \|P(x, y)\|_2 > 0$. Then $d(p, \gamma p) \geq \mu K \|(\gamma_1, \gamma_2)\|_2$ and since $\gamma \neq 1$ $(\gamma_1, \gamma_2) \neq (0, 0)$ so $d(p, \gamma p) \geq \mu K$. We have thus shown that for all $\gamma \in \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ with $\gamma \neq 1$, $d(p, \gamma p) \geq \min\{s, \mu K\} > 0$. Hence the action of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ on *Sol* is proper. Since the action is also cocompact the Svarc-Milnor Lemma says that the embedding is a quasi-isometry. □

Proposition 4. *The vertical planes in Sol given by $x = c$ or $y = c$ for some $c \in \mathbb{R}$ are isometric to the hyperbolic plane $\mathbb{H}^2$.*

Proof. Fix $c \in \mathbb{R}$ and consider the plane P given by $x = c$. The metric on *Sol* restricts to $e^{2t} dy^2 + dt^2$ on P and there is a bijection $P \to \mathbb{H}^2$ given by $(c, y, t) \mapsto (y, e^{-t})$. To see that this is an isometry consider the change of variable $t' = e^{-t}$, so as $dt' = -e^{-t} dt$. With respect to these new coordinates the metric on P is given by $e^{2t} dy^2 + dt^2 = \frac{dy^2 + (-e^{-t})^2 dt^2}{e^{-2t}} = \frac{dy^2 + dt'^2}{t'^2}$ which is the hyperbolic metric.

The case $y = c$ is similar with the isometry given by $(x, c, t) \mapsto (x, e^t)$. □

Definition 5. Let Y be a metric space and X be a subspace with the induced length metric. The *distortion function* of X in Y is $\mathrm{Dist}(n) = \sup\{d_X(a, b) | d_Y(a, b) \leq n\}$.

Definition 6. A *horocycle* in $\mathbb{H}^2$ is a subspace which, in the upper half plane model of $\mathbb{H}^2$, is either a Euclidean circle tangent to the x-axis or is a horizontal line (i.e. a Euclidean circle tangent to the boundary of $\mathbb{H}^2$ at infinity at infinity).

Lemma 7. *The distortion of a horocycle in $\mathbb{H}^2$ is $\theta(n) \sim e^n$ (i.e. $\frac{\theta(n)}{e^n} \to 1$). [No proof]*

Proposition 8. *The distortion of a horizontal plane P in Sol is $\phi(n) \simeq e^n$, where $\simeq$ is the equivalence of functions associated to Dehn functions.*

Proof. Since the metric on *Sol* is left invariant we can assume, by left translating, that P is the horizontal plane through the origin.

To prove the lower bound define $p_n = (n, 0, 0) \in Sol$ for $n \in \mathbb{R}_+$. This lies in the vertical hyperbolic plane $Q = \{(x, 0, t) | x, t \in \mathbb{R}\}$. Note that the subspace $\{(x, 0, 0) | \ \in \mathbb{R}\}$ is a horocycle in Q, so by the lemma there exists A such that $d_{Sol}(0, p_n) = d_Q(0, p_n) \leq A \log(n)$. But $d_P(0, p_n) = n$ since the metric restricted to P is $ds^2 = dx^2 + dy^2$, so P is isometric to the Euclidean plane. Thus $\theta(A \log(n)) \geq n$ and so $\theta(n) \geq e^{n/A}$.

To prove the upper bound let $p = (p_1, p_2, 0) \in P$ with $d_{Sol}(0, p) \leq n$. Say γ is a geodesic joining 0 to p in *Sol*, so $|\gamma| \leq n$. Let Q_1 and Q_2 be the vertical hyperbolic planes $\{(x, 0, t) | x, t \in \mathbb{R}\}$ and $\{(0, y, t) | y, t \in \mathbb{R}\}$ respectively. Note that under the isometries of Q_1 and Q_2 with $\mathbb{H}^2$ given in proposition 4 the subspaces $L_1 := \{(x, 0, 0) | x \in \mathbb{R}\} \subseteq Q_1$ and $L_2 := \{(0, y, 0) | y \in \mathbb{R}\} \subseteq Q_2$ both correspond to the same horocycle. Thus there exists $B \geq 1$ such that the distortion functions θ_1 and θ_2 of L_1 in Q_1 and L_2 in Q_2 respectively satisfy $\theta_1(n), \theta_2(n) \leq Be^n$. Let γ_1 and γ_2 be the projections of γ onto Q_1 and Q_2 respectively, and note that $|\gamma_1|$ and $|\gamma_2| \leq |\gamma| \leq n$. Thus $d_{Q_1}((0, 0, 0), (p_1, 0, 0)) \leq n$ and $d_{Q_2}((0, 0, 0), (0, p_2, 0)) \leq n$ and so p_1 and $p_2 \leq Be^n$. Hence $d_P(0, p) \leq \sqrt{2} Be^n$ and we have $\phi(n) \leq \sqrt{2} Be^n$. □

Definition 9. Let M be a complete Riemannian manifold. Given $c : S^1 \to M$ a null-homotopic, rectifiable loop, define the *filling area* of c to be $\mathrm{FArea}(c) = \inf\{Area(f) | f : D^2 \to M \text{ lipschitz}, \partial f = c\}$. Then the *isoperimetric function* of M

is defined to be $\mathrm{Fill}_M(l) = \sup\{\mathrm{FArea}(c) | c : S^1 \to M \text{ null} - \text{homotopic, recifiable}, |c| \leq l\}$.

Proposition 10. *Let $f : M_1 \to M_2$ be a quasi-isometry between complete Riemannian manifolds. Then $\mathrm{Fill}_{M_1} \simeq \mathrm{Fill}_{M_2}$. [No proof]*

Proposition 11. *A geodesic metric space is δ-hyperbolic if and only if it has a linear isoperimetric function. In particular $\mathrm{Fill}_{\mathbb{H}^2}$ is linear. [No proof]*

Proposition 12. *Sol has exponential isoperimetric function.*

To prove this we need the following result.

Lemma 13. *Let G be a Lie group with a left-invariant Riemannian metric and let ω be a left-invariant 2-form on G. Let $\sigma : D^2 \to G$ be an immersed 2-cell. Then there exists $K > 0$ such that $\int_\sigma \omega \leq K\mathrm{Area}(\sigma)$. [Specifically $K = \|\omega\|$.] [No proof]*

Proof of proposition 12. We only prove a lower bound.

We construct an exact left-invariant 2-form on *Sol* from the left-invariant vector fields X_1, X_2, X_3 defined above. Let X^1, X^2, X^3 be the left-invariant 1-forms dual to these, namely $e^{-t}dx$, $e^t dy$ and dt. Then let $\omega = X^1 \wedge X^2 = dx \wedge dy$. $l_g^*(X^1 \wedge X^2) = l_g^*(X^1) \wedge l_g^*(X^2) = X^1 \wedge X^2$ so this form is indeed left-invariant. $d(dx \wedge dy) = 0$ so ω is closed, and since $H^2(Sol) = H^2(\mathbb{R}^3) = 0$ ω is exact.

For $l \in \mathbb{R}_+$ we construct a loop $c_l : S^1 \to Sol$ as follows. Let $p_1 = (\frac{1}{2}l, \frac{1}{2}l, 0), p_2 = (\frac{1}{2}l, -\frac{1}{2}l, 0), p_3 = (-\frac{1}{2}l, -\frac{1}{2}l, 0), p_4 = (-\frac{1}{2}l, \frac{1}{2}l, 0)$, i.e. the points at the four corners of a square of side length l in the horizontal plane in *Sol* through 0. Let Q_i be the vertical hyperbolic plane containing p_i and p_{i+1} (addition taken modulo 4). Let γ_i be the hyperbolic geodesic in Q_i joining p_i to p_{i+1}. Let c_l be the concatenation of $\gamma_1, \dots, \gamma_4$. By lemma 7 there exits A such that $|c_l| \leq A \log(l)$ for all l. We will show that there exits $K > 0$ such that $\mathrm{FArea}(c_l) \geq Kl^2$.

By lemma 13 there exist $K > 0$ such that the area of any filling disc σ for c_l is bounded below by $K \int_\sigma \omega$. But since ω is exact the value of this integral is independent of the disc chosen, so $\mathrm{FArea}(c_l)$ is bounded below by $K \int_\sigma \omega$ for any choice of filling disc σ. We construct a choice of disc for which it is easy to evaluate the integral. Let H be the horizontal plane in *Sol* through 0. For $1 \leq i \leq 4$ let σ_i be the 2-disc in Q_i bounded by γ_i and $Q_i \cap H$. Let σ' be the 2-disc in H bounded by the four lines $Q_i \cap H$. The union of these five discs $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ' gives a filling disc for c_l consisting of a horizontal square of side length l and four vertical flaps. We now calculate $\int_\sigma \omega$. ω pulls back to 0 on each σ_i so $\int_{\sigma_i} \omega = 0$. On σ' ω is the Euclidean form $dx \wedge dy$ so $\int_{\sigma'} \omega = l^2$. Thus $\int_\sigma \omega = l^2$ and so $\mathrm{FArea}(c_l) \geq Kl^2$. Hence $\mathrm{Fill}_{Sol}(A \log(l)) \geq Kl^2$ so $\mathrm{Fill}_{Sol}(l) \geq Ke^{2l/A}$. □

Corollary 14. *Let $A \in SL_2(\mathbb{Z})$ with A conjugate in $GL_2(\mathbb{R})$ to a matrix of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for some $\lambda > 1$. Then $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ has exponential Dehn function.*

Proof. Proposition 3 showed that $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is a lattice in *Sol* and acts on it properly and cocompactly by left translation. Since the action is also free it is a covering space action and so we have that *Sol* is the universal cover of a compact manifold M with $\Pi_1(M) \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}$. By the Filling Theorem $\delta_{\mathbb{Z}^2 \rtimes_A \mathbb{Z}} \simeq \mathrm{Fill}_M \simeq \mathrm{Fill}_{\widetilde{M}} = \mathrm{Fill}_{Sol}$. □

The main result of the lectures is the following theorem on the quasi-rigidity of lattices in *Sol*.

Theorem 15. *Let Γ be a finitely generated group quasi-isometric to Sol. Then there exists $K \lhd \Gamma$ with $|K|$ finite such that $\mathbb{Z}^2 \rtimes_A \mathbb{Z} \leq_{f.i.} \Gamma/K$ for some $A \in SL_2(\mathbb{Z})$ with A conjugate in $GL_2(\mathbb{R})$ to a matrix of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for some $\lambda > 1$.*

To begin the proof of this result we require the following definitions.

Definition 16. For $t' \in \mathbb{R}$ let $P_{t'} \subseteq Sol$ be the horizontal plane $\{(x, y, t)|t = t'\}$. Then a quasi-geodesic $\gamma : \mathbb{R} \to Sol$ is said to be *vertical* if there exists M such that $\mathrm{diam}(\gamma \cap P_t) \leq M$ for all $t \in \mathbb{R}$. A *vertical quasi-geodesic ray* is defined similarly.

By a *vertical geodesic* (respectively a *vertical geodesic ray*) in Sol we mean a geodesic $\mathbb{R} \to Sol$ (respectively a geodesic ray $[0, \infty) \to Sol$) of the form $t \mapsto (x, y, t)$ or $t \mapsto (x, y, -t)$ for some $x, y \in \mathbb{R}$.

Definition 17. Let X be a metric space. Then the *Hausdorff distance* between two sets $A, B \subseteq X$ is $d_H(A, B) = \inf\{\epsilon|A \subseteq N_\epsilon(B), B \subseteq N_\epsilon(A)\}$, where N_ϵ is the ϵ-neighbourhood of a set.

Definition 18. Let X be a metric space and $\gamma_1, \gamma_2 : [0, \infty) \to X$ be geodesic rays. We say that γ_1 is *asymptotic* to γ_2 if there exists M such that $d(\gamma_1(t), \gamma_2(t)) \leq M$ for all $t \in [0, \infty)$. It can be shown that this is equivalent to the Hausdorff distance between the images of γ_1 and γ_2 being finite.

Now let $\gamma_1, \gamma_2 : [0, \infty) \to X$ be quasi-geodesic rays. We say that γ_1 and γ_2 are *asymptotic* if $d_H(\mathrm{im}(\gamma_1), \mathrm{im}(\gamma_2))$ is finite. [Note that this is not equivalent to $d(\gamma_1(t), \gamma_2(t))$ being bounded.]

We write $\gamma_1 \sim \gamma_2$ if γ_1 and γ_2 are asymptotic (quasi-)geodesic rays.

Note that quasi-isometries preserve the asymptoticity of quasi-geodesic rays.

Proposition 19. *Let γ be a vertical quasi-geodesic ray in Sol. Then there exists ϵ and a vertical geodesic ray $\overline{\gamma}$ such that $d_H(\gamma, \overline{\gamma}) \leq \epsilon$, i.e. every vertical quasi-geodesic is asymptotic to a vertical geodesic.*

Proof. ************************** □

Proposition 20. *Let $f : Sol \to Sol$ be a quasi-isometry, and let γ be a vertical quasi-geodesic ray. Then $f\gamma$ is a vertical quasi-geodesic ray.*

Proof. ************************************** □

Definition 21. Let X be a metric space. Define the *boundary* ∂X of X to be the collection of geodesic rays in X modulo the equivalence relation $\gamma_1 \sim \gamma_2$ if γ_1 and γ_2 are asymptotic.

We now introduce the following subspaces of ∂Sol. Let $\partial^U Sol$ be the collection of vertical geodesic rays (modulo asymptoticity) of the form $t \mapsto (x, y, t)$ for some $x, y \in \mathbb{R}$. Similarly define $\partial^L Sol$ to be the collection of vertical geodesic rays of the form $t \mapsto (x, y, -t)$. Note that by proposition 19 $\partial^U Sol \cup \partial^L Sol$ is equivalent to the collection of vertical quasi-geodesic rays modulo asymptoticity. Furthermore, by proposition 20, a quasi-isometry $f : Sol \to Sol$ induces a map $f_\partial : \partial^U Sol \cup \partial^L Sol \to \partial^U Sol \cup \partial^L Sol$.

Consider two vertical geodesic rays lying in the same (x, t)-plane, i.e. of the form $\gamma_1 : t \mapsto (x_1, y, t)$ and $\gamma_2 : t \mapsto (x_2, y, t)$ for some fixed $x_1, x_2, y \in \mathbb{R}$. The metric on the plane is $e^{-2t}dx^2 + dt^2$ so $d(\gamma_1(t), \gamma_2(t)) = e^{-2t} \leq 1$ for all $t \in [0, \infty)$ and hence $\gamma_1 \sim \gamma_2$. Conversely suppose that two vertical geodesic rays lie in the same (y, t)-plane, i.e. that they are of the form $\gamma_1 : t \mapsto (x, y_1, t)$ and $\gamma_2 : t \mapsto (x, y_2, t)$ for some fixed $x, y_1, y_2 \in \mathbb{R}$. Now the metric on this plane is $e^{2t}dy^2 + dt^2$ so $d(\gamma_1(t), \gamma_2(t)) = e^{2t}$ which is unbounded. Hence γ_1 and γ_2 are not asymptotic. It follows that $\partial^U Sol$ is identifiable with the collection of (x, t)-planes in Sol, which are indexed by $\mathbb{R}$, and hence is isometric to $\mathbb{R}$. Similarly $\partial^L Sol$ is also isometric to $\mathbb{R}$. Since an 'upward' and a 'downward' geodesic ray are not asymptotic, $\partial^U Sol$ and $\partial^L Sol$ are disjoint, and so their union is isometric to $\mathbb{R} \sqcup \mathbb{R}$.

Definition 22. Let X, Y be metric spaces. If $f : X \to Y$ is a quasi-isometry then there exists a quasi-isometry $f^{-1} : Y \to X$, called a *quasi-inverse* to f, such that there exists ϵ so as $d_X(x, f^{-1}f(x)) \leq \epsilon$ for all $x \in X$ and $d_Y(y, ff^{-1}(y)) \leq \epsilon$ for all $y \in Y$.

Let $\sim$ be the equivalence relation on the set of quasi-isometries $X \to X$ given by $f \sim g$ if there exists ϵ such that $d(f(x), g(x)) \leq \epsilon$ for all $x \in X$. Then the *quasi-isometry group* of X, written $\mathcal{QI}(X)$, is defined to be this set of quasi-isometries modulo $\sim$.

We have shown that a quasi-isometry $Sol \to Sol$ induces a map $f_\partial : \mathbb{R} \sqcup \mathbb{R} \to \mathbb{R} \sqcup \mathbb{R}$. Note that if f_1, f_2 are quasi-isometries $Sol \to Sol$ with $f_1 \sim f_2$ and γ is a vertical quasi-geodesic then $f_1(\gamma)$ and $f_2(\gamma)$ are asymptotic, so in fact each $f \in \mathcal{QI}(Sol)$ induces a map $f_\partial : \mathbb{R} \sqcup \mathbb{R} \to \mathbb{R} \sqcup \mathbb{R}$. ********We can show that f_∂ is continuous.************** Now let f^{-1} be the quasi-inverse to $f \in \mathcal{QI}(Sol)$ and let γ be a vertical quasi-geodesic. Then there exists ϵ such that $d(ff^{-1}(x), x) \leq \epsilon$ and $d(f^{-1}f(x), x) \leq \epsilon$ for all $x \in Sol$. Thus $f^{-1}f(\gamma) \sim \gamma \sim ff^{-1}(\gamma)$ and so $(f^{-1})_\partial$ is an inverse to f_∂ in $\mathcal{C}(\mathbb{R})$. f_∂ is therefore a homeomorphism of $\mathbb{R} \sqcup \mathbb{R}$, and we have shown that there exist a homomorphism $\mathcal{QI}(Sol) \to \mathrm{Homeo}(\mathbb{R} \sqcup \mathbb{R})$.

Definition 23. A *quasi-action* of a finitely generated group G on a metric space X is a homomorphism $G \to \mathcal{QI}(X)$.

Lemma 24. *Let Γ be a finitely generated group quasi-isometric to a metric space X. Then there exists a quasi-action $\gamma \mapsto q_\gamma$ of Γ on X. Furthermore there exists $\lambda \geq 1$ and $\epsilon \geq 0$ such that q_γ is a (λ, ϵ)-quasi-isometry for all $\gamma \in \Gamma$.*

Proof. Let $\theta : \Gamma \to X$ be the hypothesized quasi-isometry. Then for $\gamma \in \Gamma$ let $q_\gamma = \theta L_\gamma \theta^{-1}$ where $L_\gamma : \Gamma \to \Gamma$ is left multiplication by γ. Since L_γ is an isometry, and hence trivially a quasi-isometry, $q_\gamma \in \mathcal{QI}(X)$. If $\gamma_1, \gamma_2 \in \Gamma$ then $q_{\gamma_1\gamma_2} = fL_{\gamma_1\gamma_2}f^{-1} = fL_{\gamma_1}L_{\gamma_2}f^{-1} \sim fL_{\gamma_1}f^{-1}fL_{\gamma_2}f^{-1} = q_{\gamma_1}q_{\gamma_2}$ so we do indeed have a quasi-action. If f^{-1} is a (λ_1, ϵ_1)-quasi-isometry and f is a (λ_2, ϵ_2)-quasi-isometry then q_γ is a $(\lambda_2\lambda_1, \lambda_2\epsilon_1 + \epsilon_2)$-quasi-isometry for each γ. $\square$

Applying this construction to the quasi-isometry given in the hypothesis of the theorem, we obtain quasi-action of Γ on Sol which can be shown to have the following two properties:

- **Co-compactness:** There exists $C \in \mathbb{R}$ such that for all $x \in Sol$ there exists $\gamma \in \Gamma$ with $d(x, q_\gamma(0)) \leq C$, where 0 is the point $(0,0,0) \in Sol$.
- **Proper discontinuity:** For all $x \in Sol$ and for all $C \geq 0$, $|\{\gamma \in \Gamma | d(q_\gamma(x), x) \leq C\}|$ is finite.

We showed above that there exists a homomorphism $\mathcal{QI}(Sol) \to \mathrm{Homeo}(\mathbb{R} \sqcup \mathbb{R})$, and so by composition we have a homomorphism $\Psi : \Gamma \to \mathrm{Homeo}(\mathbb{R} \sqcup \mathbb{R})$. Now, by passing to an index 2 subgroup if necessary (which will not affect the conclusion of the theorem), we can assume that $\Psi(\gamma)$ fixes each component of $\mathbb{R} \sqcup \mathbb{R}$ for each $\gamma \in \Gamma$. Thus we have $\Psi : \Gamma \to \mathrm{Homeo}(\mathbb{R})^2$.

Definition 25. $f \in \mathrm{Homeo}(\mathbb{R})$ is a *quasi-symmetric* homeomorphism if there exists $K \geq 1$ such that for all $x, y \in \mathbb{R}$

$$K^{-1} \leq \frac{|f(x) - f(z)|}{|f(x) - f(y)|} \leq K$$

where z is the midpoint of x and y.

A subgroup $H \leq \mathrm{Homeo}(\mathbb{R})$ of quasi-symmetric homeomorphisms is *uniform* if there is such a constant K which holds for all $h \in H$.

Proposition 26. *Let $f : Sol \to Sol$ be a (λ, ϵ)-quasi-isometry, and let $(f^1_\partial, f^2_\partial) \in \mathrm{Homeo}(\mathbb{R})^2$ be the induced homeomorphism. Then f^1_∂ and f^2_∂ are quasi-symmetric with constant depending only on λ and ϵ.*

Proof. ********************** □

It follows that the image of Γ in $\mathrm{Homeo}(\mathbb{R})^2$ is the direct product of two uniform groups of quasi-symmetric homeomorphisms.

Theorem 27 (Hinkkanen)**.** *If $H \leq \mathrm{Homeo}(\mathbb{R})$ is a uniform group of quasi-symmetric homeomorphisms then there exists a quasi-symmetric homeomorphism $\rho \in \mathrm{Homeo}(\mathbb{R})$ such that $\rho H \rho^{-1} \leq \mathrm{Aff}(\mathbb{R})$. [No proof]*

Thus by composing with a conjugation we can assume that $\Psi : \Gamma \to \mathrm{Aff}(\mathbb{R})^2$.

Lemma 28. $|\ker(\Psi)|$ *is finite.*

Proof. Ψ is constructed as a composition of homomorphisms

$$\Gamma \to \mathcal{QI}(Sol) \to \mathrm{Homeo}(\mathbb{R})^2 \to \mathrm{Homeo}(\mathbb{R})^2.$$

The last of these, since it is a conjugation, is an isomorphism, so it suffices to prove that the composition of the first two homomorphisms has finite kernel.

Let $g \in \ker \Psi$. Fix a vertical hyperbolic plane $\mathbb{H}^2 \subseteq Sol$, and choose a point $c \in \mathbb{H}^2$ and two geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to \mathbb{H}^2$ with $\gamma_i(0) = c$ and γ_1 and γ_2 representing different boundary points in $\partial\mathbb{H}^2$. Since $g \in \ker \Psi$, q_g fixes the boundary of *Sol* and hence of $\mathbb{H}^2$, so there exists K such that $d_H(q_g\gamma_1, \gamma_1) \leq K$ and $d_H(q_g\gamma_2, \gamma_2) \leq K$. Hence there exist x_1 and x_2 such that $d(q_g\gamma_1(0), \gamma_1(x_1)) \leq K$ and $d(q_g\gamma_2(0), \gamma_2(x_2)) \leq K$, and thus $d(\gamma_1(x_1), \gamma_2(x_2)) \leq 2K$. Choose $M \in \mathbb{R}$ such that $g(\gamma_1(x), \gamma_2(y)) > 2K$ for all $x, y \geq M$. Then $x_1, x_2 < M$. Thus $d(q_g\gamma_1(0), \gamma(0)) \leq d(q_g\gamma_1(0), \gamma_1(x_1)) + d(\gamma_1(x_1), \gamma_1(0)) \leq K + M$. But the quasi-action of Γ on *Sol* is properly-discontinuous, so there are only a finite number of $g \in \Gamma$ which satisfy such an inequality. □

Thus we have that $\Gamma/K \leq \mathrm{Aff}(\mathbb{R})^2$ for some finite K. Since $\mathrm{Aff}(\mathbb{R})$ is soluble so is $\mathrm{Aff}(\mathbb{R})^2$ and hence Γ/K.

Definition 29. A group G is a *Poincaré duality* group if it has a finite dimensional $K(G, 1)$, say of dimension n, and $H_i(G) \cong H^{n-i}(G)$ for $0 \leq i \leq n$.

Theorem 30. *Let G be a soluble Poincaré duality group. Then G is virtually polycyclic. [No proof]*

Theorem 31 (Gerston)**.** *The Poincaré duality property of groups is an invariant of quasi-isometry. [No proof]*

We want to show that Γ/K has the poincaré duality property. To see this choose $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, say, so as $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is quasi-isometric to *Sol*, and hence to Γ, and hence, since $|K|$ is finite, to Γ/K. The action of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ on *Sol* by left translation is a covering space action, and *Sol* is simply connected, so by quotienting we obtain a $K(\mathbb{Z}^2 \rtimes_A \mathbb{Z}, 1)$ space M. Since *Sol* is a 3-manifold so is M, and so Poincaré duality gives that $H_i(M) \cong H^{3-i}(M)$ for $0 \leq i \leq 3$. It follows that $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ has the Poincaré duality property, and hence by theorem 31 so does Γ/K. We deduce that Γ/K is virtually polycyclic.*****************Now consider Hirsch length and growth rates. This complete the proof of theorem 15.