Paulin's Theorem

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1 Hyperbolic Metric Spaces

The notion of a hyperbolic metric space is an attempt to capture the idea of negative curvature in the setting of general metric spaces. There are many equivalent definitions, but the following one, in the context of geodesic metric spaces, is perhaps the most intuitive.

Definition 1.1 (Slim triangles and hyperbolic metric spaces) *Let X be a geodesic metric space and $\Delta \subset X$ a geodesic triangle; that is, a set of three geodesic segments in X such that any pair of segments shares precisely one endpoint. Then Δ is δ-**slim** if any side of Δ is contained in the δ-neighbourhood of the other two. The metric space X is δ-**hyperbolic** if every triangle is δ-slim, and X is called **hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$.*

Note that the Euclidean plain is clearly not hyperbolic: just consider right-angled isosceles triangles of arbitrary size. The most trivial example of a hyperbolic space is if X is any metric tree. Since all triangles in a tree are tripods, such an X is 0-hyperbolic. It can be useful to think of hyperbolic spaces as thickened trees.

The next example explains why the term hyperbolic is used.

Example 1.2 (The hyperbolic plane) *Let $X = \mathbb{H}^2$. Consider a geodesic triangle Δ with sides α, β, γ. For a point $x \in \alpha$, the shortest distance to $\beta \cup \gamma$ is precisely the radius of the semicircle inscribed in Δ centered at x. So Δ is δ-slim, where δ is the radius of the largest semicircle inscribed in Δ. Since the area of triangles in $\mathbb{H}^2$ is bounded above by π it follows that the radii of inscribed semicircles is also bounded, so $\mathbb{H}^2$ is hyperbolic. The best δ can be obtained by calculating the radius of the semicircle inscribed in an ideal triangle, giving the result $\delta = \frac{1}{2}\log(3 + 2\sqrt{2})$.*

Recall the basic definitions of coarse geometry. That is that, for $\lambda \geq 1$ and $\epsilon \geq 0$, a (not necessarily continuous) map of metric spaces $f : X \to Y$ is a (λ, ϵ)**-quasi-isometric embedding** if

$$\frac{1}{\lambda} d_X(x_1, x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \epsilon$$

for all $x_1, x_2 \in X$. If also there exists $C \geq 0$ such that for all $y \in Y$, $d_Y(y, \mathrm{im}(X)) \leq C$ then f is a **quasi-isometry**. In this case X and Y are called **quasi-isometric**; using the axiom of choice it is easy to construct a (λ, ϵ)-quasi-isometry $g : Y \to X$, so being quasi-isometric is an equivalence relation. A **quasi-geodesic** in X is a quasi-isometric embedding of an interval into X.

The fundamental theorem concerning hyperbolic metric spaces asserts the stability of quasi-geodesics.

Theorem 1.3 *For all $\delta \geq 0$, $\lambda \geq 1$,$\epsilon \geq 0$ there exists a constant R with the following property. If X is a δ-hyperbolic geodesic metric space, α is a quasi-geodesic segment in X and β is a geodesic between the end-points of α then the Hausdorff distance between the images of α and β is at most R.*

It follows immediately that the term 'hyperbolic' can be defined just as well in terms of (λ, ϵ)-quasi-geodesic triangles instead of geodesic triangles. Hence, among geodesic metric spaces, the property of being hyperbolic is an invariant of coarse geometry.

2 Hyperbolic groups

The first step in geometric group theory is to turn groups into geometric objects. In the case of finitely generated groups, an easy way to do this is via the group's Cayley graph.

Definition 2.1 (Cayley Graph) *Let G be a finitely generated group. Fix a finite generating set $S \subset G$. Then the **Cayley Graph** of G with respect to S, denoted $C_S(G)$, is the graph with vertex set G and a single edge between the unordered pair $g, h \in G$ if and only if, for some $s \in S$, $sg = h$. It can be given a metric by setting the length of each edge to 1 and insisting that the metric be geodesic.*

Note that a Cayley graph is connected and regular. Loops in the Cayley graph correspond to relations in the generators.

Example 2.2 *If G is a free group then, as there are no relations between the generators, there are no loops in the Cayley graph. So it is a regular tree.*

The group G embeds naturally into the Cayley graph, and the induced metric on G is denoted d_S and called the *word metric.* This embedding is a quasi-isometry (set $\lambda = 1$, $\epsilon = 0$ and $C = \frac{1}{2}$). Note also, that the action of the group on itself by right-multiplication extends to an isometric action on the Cayley graph.

If a different finite generating set, $T \subset G$, is chosen instead, then an easy induction argument shows that the identity map

$$(G, d_S) \to (G, d_T)$$

is a quasi-isometric embedding, with $\lambda = 2\max_{t\in T} d_S(e,t)$ and $\epsilon = 0$. Therefore the Cayley graph of a finitely generated group is well-defined up to quasi-isometry.

Definition 2.3 *A finitely generated group is **(word) hyperbolic** if its Cayley graph is a hyperbolic metric space.*

The simplest example of a hyperbolic group is any finitely generated free group: as noted above its Cayley graph is a tree. Hyperbolic groups are often candidates for generalizing results that apply to free groups.

Recall the Švarc-Milnor Lemma, also known as the Fundamental Theorem of Geometric Group Theory.

Theorem 2.4 (The Švarc-Milnor Lemma) *Let X be a length space on which a group G acts properly and cocompactly by isometries. Then G is finitely generated and for any choice of base-point $x_0 \in X$, the map $G \to X$ given by $g \to g.x$ is a quasi-isometry.*

This gives a much larger class of examples.

Example 2.5 (Surface groups are hyperbolic) *Let Σ be any surface of Euler characteristic less than 0, possibly with open discs removed. Then its universal cover carries a natural metric as a convex subset of the hyperbolic plane. Therefore surface groups act properly and cocompactly on a hyperbolic space, and so are hyperbolic.*

3 $\mathbb{R}$-trees and group actions

$\mathbb{R}$-trees are defined to be 0-hyperbolic spaces. That is to say, any triangle is a tripod. So the simplest example of an $\mathbb{R}$-tree is just any metric tree. In this context ordinary trees, are called *simplicial*. However, in general they can be much more badly behaved.

Example 3.1 (The SNCF metric) *Let $X = \mathbb{R}^2$, but with the following metric. The distance between (x_1, y_1) and (x_2, y_2) is defined to be $|y_1 - y_2|$ if $x_1 = x_2$, and $|y_1| + |x_1 - x_2| + |y_2|$ otherwise. This space is an $\mathbb{R}$-tree, but cannot be given a simplicial structure. For a separable example, just consider the subtree in which all the 'branches' have rational x-coordinates.*

Bass and Serre developed a comprehensive theory of group actions on simplicial trees. They described the quotient as a *graph of groups*, consisting of a finite graph with groups corresponding to the vertices and edges, and monomorphism from edge groups into the initial and terminal vertex groups. The simplest cases, out of which all others are built, are *amalgamated free products* and *HNN-extensions*.

Definition 3.2 *Let $G = \langle A \big| R \rangle$, $H = \langle B \big| S \rangle$ be finitely-generated groups. Let K be a subgroup of G, and $\phi : K \to H$ a monomorphism. Then the* ***amalgamated free product of** G **and** H **over** ϕ* *is defined to be*

$$G \star_\phi H = \langle A, B \big| R, S, \{g^{-1}\phi(g) | g \in K\} \rangle;$$

it is often abusively denoted $G \star_K H$.

Now suppose ϕ is a monomorphism $K \to G$. Then the ***HNN-extension of** G **over** ϕ* *is defined to be*

$$G\star_\phi = \langle A, \{t\} \big| R, \{t^{-1}g^{-1}t\phi(g) | g \in K\} \rangle,$$

often abusively denoted $G\star_K$.

Amalgamated free products correspond to the case of two vertices joined by one edge, and HNN-extensions correspond to a single vertex with a loop.

As a result of the work of Rips, actions on $\mathbb{R}$-trees are also well understood. For example, there is the following theorem.

Theorem 3.3 *Suppose a finitely presentable hyperbolic group with one end acts on an $\mathbb{R}$-tree with no global fixed points and virtually cyclic arc stabilizers. Then the group splits as an amalgamated free product or HNN-extension.*

4 Degenerations of group actions on hyperbolic spaces

This understanding of group actions on $\mathbb{R}$-trees suggests a strategy for proving theorems about groups that act on hyperbolic spaces, as follows. Take sequence of actions on a hyperbolic space. If the 'scale' of the action increases, then they should degenerate to an action on a 0-hyperbolic space. This requires a notion of convergence of group actions.

Definition 4.1 (Convergence of based G-spaces) *Let G be a finitely-generated group. A* ***based** G**-space*** *is a triple (X, x, ρ), where X is a metric space, $x \in X$ is a point and ρ is a homomorphism $G \to \mathrm{Isom}(X)$. This induces a pseudo-metric on G, given by*

$$d_{(X,x,\rho)}(g, h) = d_X(\rho(g)x, \rho(h)x)$$

for $g, h \in G$, which has an image in $\mathcal{PED}(G)$, the projectivized space of non-zero G-equivariant pseudo-metrics on G, equipped with the compact-open topology. For (X_i, x_i, ρ_i) a sequence of based G-spaces, write

$$(X_i, x_i, \rho_i) \to (X, x, \rho)$$

if $d_{(X_i,x_i,\rho_i)}$ converges to $d_{(X,x,\rho)}$ in $\mathcal{PED}(G)$.

Lemma 4.2 *$\mathcal{PED}(G)$ is compact.*

Proof: Fix S a finite generating set for G. Let d_i be a sequence of equivariant pseudo-metrics on G. Without loss of generality, for all $s \in S$, $d_i(e, s) \leq 1$.

Passing to successive subsequences ensures, at the nth stage, that for all words w of length at most n, $d(e, w)$ converges (since by induction, for such a word w, $d(e, w) \leq n$). Now a diagonal argument provides a subsequence for which $d_i(e, g)$ converges for all g, to $d(e, g)$ say. This propagates to all distances by G-equivariance. *QED*

A notion of scale for a group action is also needed. Fix a finite generating set S for G. The *scale function* for (X, x, ρ) is defined by

$$\sigma_S(y) = \max_{s \in S} d(x, \rho(s)x)$$

for $y \in X$.

Now the notion of degenerating actions on hyperbolic spaces is encompassed in a compactness theorem.

Theorem 4.3 (Compactness Theorem for actions on hyperbolic spaces) *Let X be a cocompact hyperbolic metric space and $\rho_i : G \to \mathrm{Isom}(X)$ a sequence of homomorphisms. Then, after passing to a subsequence, one of the following holds.*

1. *There exist isometries $\phi_i \in \mathrm{Isom}(X)$ such that the sequence $\phi_i \circ \rho \circ \phi_i^{-1}$ converges in the compact-open topology to a representation $\rho : G \to \mathrm{Isom}(X)$.*

2. *The sequence of based G-spaces (X_i, x_i, ρ_i) converges to an action of G on an $\mathbb{R}$-tree.*

Proof: Fix a generating set S for G, and let $\sigma_{S,i}$ be the scale function for (X_i, x_i, ρ_i).

1. Suppose $\sigma_{S,i}$ does not converge to infinity. Then by passing to a subsequence, the $d_{(X_i, x_i, \rho_i)}$ are uniformly bounded. By the cocompactness of X there exist isometries ϕ_i of X such that $\phi_i(x_i)$ lies in some compact subset K for all i. Now apply the Arzela-Ascoli theorem.

2. Now suppose $\sigma_{S,i}$ converges to infinity. By the last lemma, passing to a subsequence there does exist some limit. It is clear that, for a δ-hyperbolic space, if the metric is multiplies by a positive constant then the optimal δ is divided by it. So the limiting pseudo-metric on G is 0-hyperbolic (in some sense for non-geodesic spaces). There is a slight technical problem here, to 'sew together' a non-geodesic 0-hyperbolic place to give an $\mathbb{R}$-tree, but it is not difficult to resolve.

QED

For this theorem to be of any use, it is necessary to ensure that the resulting action on an $\mathbb{R}$-tree is non-trivial. This comes down to correct selection of the base-point. In fact, if the x_i are chosen to minimize $\sigma_{S,i}$ on X, then the resulting action on an $\mathbb{R}$-tree also has no global fixed points. Here are the details. Let $F \subset G$ be a finite subset. Then for $x \in T$, define $X_i(F, x)$ as follows. Whenever $g, h \in F$ are such that x lies on the geodesic between them, let $X_i(F, x)$ contain every point in X_i on a geodesic between $\rho_i(g)x_i$ and $\rho_i(h)x_i$ that divides the geodesic in the same ratio as x divides the geodesic in T. These have the following properties.

Proposition 4.4 *1. For all x there always exists finite F such that $X_i(F, x)$ is non-empty.*

2. $\rho_i(g)X_i(F, x) = X_i(gF, \rho(g)x)$.

3. If $F \subseteq F'$ then $X_i(F, x) \subseteq X_i(F', x)$.

4. $\frac{\mathrm{diam} X_i(F,x)}{\sigma_{S,i}(x_i)} \to 0$ as $i \to \infty$.

Now suppose $x \in T$ is a global fixed point. Then for any $y_i \in X_i(F, x)$ it follows that $y_i, \rho_i(g)y_i \in X(gF \cup F, x)$, in which case assertion 4 of the above proposition contradicts the choice of x_i.

5 Paulin's Theorem

In this section the ideas developed so far are applied to prove a group-theoretic result.

Definition 5.1 *Let G be a finitely-generated group. Then an* ***inner automorphism*** *of G is an automorphism of the form $x \to g^{-1}xg$ for some $g \in G$. The group of inner automorphisms of G is denoted $Inn(G)$. Now the* ***outer automorphism group*** *of G is defined to be*

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G).$$

Incidentally, there exists a more geometric interpretation of G, as a result of a theorem of Nielsen.

Theorem 5.2 *Let Σ be a surface. Then*

$$\mathrm{Out}(\pi_1(\Sigma)) \cong \mathrm{MCG}(\Sigma).$$

The aim of this section is to prove the following theorem.

Theorem 5.3 (Paulin, Bridson and Swarup) *Let G be a 'generic' finitely-presentable hyperbolic group with $\mathrm{Out}(G)$ infinite. Then G splits as an amalgamated free product or HNN-extension.*

By the theorem on group actions on $\mathbb{R}$-trees discussed earlier, this follows from the next theorem.

Theorem 5.4 *Let G be a finitely-generated hyperbolic group with* $\mathrm{Out}(G)$ *infinite. Then G acts on an $\mathbb{R}$-tree with no global fixed points and virtually cyclic arc stabilizers.*

So the theory developed above is employed. Fix a finite generating set S of G, and let X be the corresponding Cayley graph of G. By assumption, Γ is δ-hyperbolic for some δ. Because $\mathrm{Out}(G)$ is infinite, it is possible to choose an infinite sequence of $\phi_i \in \mathrm{Aut}(G)$, none of which are inner automorphisms, and no two of which have identical images in $\mathrm{Out}(G)$. Define the action ρ_i of G on X by stipulating that $g \in G$ acts via $\phi_i(g)$. Now the compactness theorem can be applied.

Note that $\sigma_{S,i}$ is integer-valued at vertices and edge mid-points of X, and linear in between, so base-points can certainly be chosen to minimize $\sigma_{S,i}$. Therefore any resulting action on an $\mathbb{R}$-tree won't have global fixed points.

Suppose $\sigma_{S,i}(x_i)$ is bounded, say by M. Then for y_i the nearest vertex to x_i and all $s \in S$ it follows that $d(e, y_i^{-1}\phi_i(s)y_i) = d(y_i, \phi_i(s)y_i) \leq M+2$. Therefore two of the ϕ_i have the same image in $\mathrm{Out}(G)$, contradicting their choice.

Therefore G acts without global fixed points on an $\mathbb{R}$-tree. It remains to show that arc stabilizers are virtually cyclic. This requires a fiddly argument using hyperbolicity, which is too lengthy to detail here, but there is one neat simplification worth mentioning.

Lemma 5.5 *Let G be a virtually abelian hyperbolic group. Then G is virtually cyclic.*

Proof: Let $A \subseteq G$ be an abelian subgroup of finite index. It acts properly and cocompactly on G, so by the Švarc-Milnor Lemma is finitely-generated and quasi-isomorphic to G; therefore it is hyperbolic. Suppose it is of rank 2 or more. Then its Cayley graph contains an isometric copy of a 2-dimensional Euclidean lattice. This contradicts the fact that it is hyperbolic. *QED*

References

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