# Mostow Rigidity

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## 0 Introduction

### Lie Groups and Symmetric Spaces

We will be concerned with

(a) semi-simple Lie groups with trivial centre and no compact factors and

(b) simply connected, non-positively curved symmetric spaces of non-compact type.

There is a bijection between these two classes of object. Given a Lie group $G$ of class (a) one endows it with a left-invariant Riemannian metric, takes a maximal compact subgroup $K$, and forms the quotient space $G/K$. This manifold, together with the inherited Riemannian metric, is a symmetric space of class (b). Conversely, given a symmetric space $M$ of class (b), the connected component of the identity in $\text{Isom}(M)$ is a Lie group of class (a).

**Definition 1.** The (real) *rank* of a simply connected, non-positively curved symmetric space is the maximal dimension of an embedded Euclidean subspace.

The (real) *rank* of a semi-simple Lie group (of class (a)) is the rank of its associated symmetric space.

There is a fundamental dichotomy between the rank 1 and higher rank (i.e. rank $\geq 2$) cases.

### Examples

**Rank** 1 All rank 1 symmetric spaces of class (b) are of the form $\mathbb{K}H^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (the quaternions) except for one exceptional example of dimension 8 based on the octonions.

**Rank** $\geq 2$ In higher rank one sees much more diversity. The key example is the Lie group $SL(n, \mathbb{R})$ and the associated symmetric space $\frac{SL(n,\mathbb{R})}{O(n,\mathbb{R})}$.

## Mostow Rigidity

**Theorem 2** (Mostow Rigidity)**.** *Let $G$ be a Lie group of class (a), and let $\Gamma_1$ and $\Gamma_2$ be lattices in $G$ with $\Gamma_1 \cong \Gamma_2$ (as abstract groups). If $rk(G) = 1$ we require that $G \ncong \mathrm{Isom}_+(\mathbb{H}^2) \cong PSL(2,\mathbb{R})$. If $rk(G) \geq 2$ we require that the $\Gamma_1$ and $\Gamma_2$ are irreducible. Then there exists $g \in G$ such that $\Gamma_1^g = \Gamma_2$.*

The proof in the higher rank case is completely different to that in the rank 1 case. The higher rank case leads to super-rigidity and arithmeticity, i.e. that all irreducible lattices in higher rank Lie groups arise from arithmetic constructions.

The aim in this series of lectures is to prove Mostow rigidity in the real hyperbolic rank 1 case, i.e. where $G = SO(n,1) = \mathrm{Isom}_+(\mathbb{H}^n)$ for $n \geq 3$. We thus aim to prove:

**Theorem 3.** *Let $\Gamma_1$ and $\Gamma_2$ be lattices in $SO(n,1)$ for $n \geq 3$. If $\Gamma_1 \cong \Gamma_2$ then there exists $g \in SO(n,1)$ such that $\Gamma_1^g = \Gamma_2$.*

**Corollary 4.** *In the torsion free case we have that if $M_1 \cong \mathbb{H}^n/\Gamma_1$ and $M_2 \cong \mathbb{H}^n/\Gamma_2$ are finite volume hyperbolic manifolds then the following are equivalent:*

1. $\Pi_1 M_1 \cong \Pi_1 M_2$
2. *$M_1$ is homotopy equivalent to $M_2$*
3. *$M_1$ is homeomorphic to $M_2$*
4. *$M_1$ is isometric to $M_2$.*

## Structure of the Course

(i) Some real hyperbolic geometry.

(ii) Some possible strategies of proof.

(iii) Then break into two cases:

**Co-compact case** where $\Gamma_1$, $\Gamma_2$ are uniform lattices and $M_1$, $M_2$ are compact. In this case we give as detailed as possible a version of the Gromov–Thurston proof, which contains a minimal amount of analysis.

**Non-compact case** where $\Gamma_1$, $\Gamma_2$ are non-uniform lattices and $M_1$, $M_2$ have finite volume but are not compact. In this case we consider the quasi-isometric rigidity given by the following result.

**Theorem 5.** *Let $\Gamma$ be a non-uniform lattice in $SO(n,1)$ for $n \geq 3$. Let $\Lambda$ be a finitely generated group quasi-isometric to $\Gamma$. Then there exists a short exact sequence of groups*

$$1 \to K \to \Lambda \to \Gamma' \to 1$$

*with $|K|$ finite and $\Gamma'$ a lattice in $SO(n,1)$ such that $\Gamma$ and $\Gamma'$ are commensurable (i.e. have a common subgroup of finite index).*

**Corollary** (of theorems 1 and 2)**.** *Let $\Gamma_1$, $\Gamma_2$ be lattices in $SO(n,1)$ for $n \geq 3$. Then $\Gamma_1$ is quasi-isometric $\Gamma_2$ if and only if either*

- *$\Gamma_1$ and $\Gamma_2$ are co-compact or*
- *$\Gamma_1$ and $\Gamma_2$ are not co-compact and there exists $g \in SO(n,1)$ such that $\Gamma_1 \cap \Gamma_2^g$ has finite index in both $\Gamma_1$ and $\Gamma_2^g$.*

[Note to reader: This course structure does not appear to have been followed too closely.]

# 1 Basic Geometry of $\mathbb{H}^n$

See Milnor - First 150 years of hyperbolic geometry, Bull. AMS 1982.

## Hyperboloid Model

**Definition 6.** $\mathbb{E}^{n,1}$ is $\mathbb{R}^{n+1}$ with the symmetric form of type $(n,1)$ given by

$$\langle \underline{x}, \underline{y} \rangle = \sum_{r=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

for $\underline{x} = (x_1, \ldots, x_{n+1})$ and $\underline{y} = (y_1, \ldots, y_{n+1})$. $S_+$ and $S_-$ are the upper and lower components respectively of the sphere of radius $-1$ in $\mathbb{E}^{n,1}$, i.e.

$$S_+ = \{\underline{x} \in \mathbb{E}^{n,1} | \langle \underline{x}, \underline{x} \rangle = -1, x_{n+1} > 0\}$$

and

$$S_- = \{\underline{x} \in \mathbb{E}^{n,1} | \langle \underline{x}, \underline{x} \rangle = -1, x_{n+1} < 0\}.$$

$\mathbb{H}^n$ is $S_+$ with the Riemannian metric $\langle\ ,\ \rangle|_{TS_+}$.

The following lemma shows that this metric on $\mathbb{H}^n$ is a genuine Riemannian metric.

**Lemma 7.** *The inner product $\langle\ ,\ \rangle$ is positive definite when restricted to $TS_+$.*

The geodesics in this model of $\mathbb{H}^n$ are then easily seen to be the intersections of $S_+$ with the dimension 2 hyperplanes in $\mathbb{E}^{n,1}$.

**Definition 8.** $O(n,1) = \{A \in Mat_{n+1,n+1}(\mathbb{R}) | \langle A\underline{x}, A\underline{y} \rangle = \langle \underline{x}, \underline{y} \rangle \forall x, y \in \mathbb{E}^{n,1}\}$. Thus $O(n,1)$ preserves $S_+ \cup S_-$, but some elements interchange the two sheets. $SO(n,1) = \{A \in O(n,1) | \det A = 1\}$ is the index 2 subgroup in $O(n,1)$ which preserves $S_+$.

**Lemma 9.** *$SO(n,1)$ acts transitively on $S_+ = \mathbb{H}^n$ with point stabilizer $SO(n)$.*

The drawbacks of the hyperboloid model of $\mathbb{H}^n$ include the facts that:

(i) It is difficult to check the homogeneity of $\mathbb{H}^n$.

(ii) One doesn't get a nice description of the Riemannian metric.

(iii) The boundary at infinity is not obvious.

We solve these problems by finding different models of $\mathbb{H}^n$.

## Klein/Beltrami Model

**Definition 10.** $B^n = \{(x_1, \ldots, x_n, 1) \in \mathbb{E}^{n,1} | x_1^2 + \ldots + x_n^2 < 1\}$. The Riemannian metric on $B^n$ is obtained by pulling back the metric on $S_+$ via the bijection $S_+ \to B^n$ given by projecting from the origin.

It is easy to see, via this map, that the geodesics in the Beltrami model are the intersections of Euclidean straight lines with the disc $B^n$. Thus $\mathbb{H}^n$ is uniquely geodesic.

## Hemisphere Model

**Definition 11.** $J^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{E}^{n,1} | x_1^2 + \ldots, +x_{n+1}^2 = 1, x_{n+1} > 0\}$. The Riemannian metric on $J^n$ is obtained by pulling back the metric on $B^n$ via the bijection $B^n \to J^n$ given by vertical projection.

The geodesics in the hemisphere model are the intersection of the hemisphere $J^n$ with vertical planes. They are thus semicircles orthogonal to $\partial J^n$.

## Poincaré Disc Model

**Definition 12.** $D^n = \{(x_1, \ldots, x_n, 0) \in \mathbb{E}^{n,1} | x_1^2 + \ldots + x_n^2 < 1\}$. The Riemannian metric on $D^n$ is obtained by pulling back the metric on $J^n$ via the bijection $J^n \to D^n$ given by stereographic projection from $(0, \ldots, 0, -1)$.

Since stereographic projection is conformal the geodesics in $D^n$ are semicircles orthogonal to $\partial D^n$.

**Lemma 13.** *The Riemannian metric obtained on $D^n$ is*

$$ds_{\mathbb{H}}^2(x) = \frac{4ds_{\mathbb{E}}^2(x)}{(1 - \|x\|_{\mathbb{E}}^2)^2}.$$

## Upper Half-Space Model

**Definition 14.** $U^n = \{(1, x_2, \ldots, x_{n+1}) \in \mathbb{E}^{n,1} | x_{n+1} > 0\}$. The Riemannian metric on $U^n$ is obtained by pulling back the metric on $D^n$ via the bijection $B^n \to J^n$ given by the Cayley transform.

**Lemma 15.** *The Riemannian metric this gives on $U^n$ is*

$$ds_{\mathbb{H}}^2(\underline{x}) = \frac{ds_{\mathbb{E}}^2(\underline{x})}{x_{n+1}^2}.$$

In this model it is easy to see that horospheres in $\mathbb{H}^n$ have a Euclidean structure. Indeed let $p$ be the point where the horosphere touches the horizontal plane $x_{n+1} = 0$. Map $p$ to infinity so as the horosphere becomes a horizontal plane $x_{n+1} = h$, for some $h \in \mathbb{R}$. The restriction of the hyperbolic metric to this plane is just $\frac{ds_{\mathbb{E}}}{h}$.

NB: This shows that parabolic subgroups of lattices in $SO(n, 1)$ are Bieberbach groups, i.e. have a subgroup $\mathbb{Z}^{n-1}$ of finite index.

### Boundary at Infinity

In $D^n$, the Poincaré disc model, the (Euclidean) unit sphere appears as the set of ideal points $\overline{D^n} \setminus D^n$. This is apparently an artifact of the embedding $D^n \hookrightarrow \mathbb{E}^n$, but in fact $\partial\mathbb{H}^n$ is a sphere of dimension $n-1$ for intrinsic reasons and plays a central role in rigidity theory.

### Ideal Simplices

Consider a geodesic triangle $\Delta$ in $\mathbb{H}^2$ with interior angles $\alpha, \beta, \gamma$ at its vertices. The Gauss–Bonnet Theorem shows that $\text{Area}(\Delta) = \pi - \alpha - \beta - \gamma$. Thus an ideal triangle in $\mathbb{H}^2$ has area $\pi$. It can be shown that up to the action of $SO(2,1)$ there is a unique ideal triangle in $\mathbb{H}^2$.

Exercises on the higher dimension case:

(i) Up to the action of $SO(n,1)$ there is a unique regular $n$-simplex in $\mathbb{H}^n$, where by regular we mean all dihedral angles equal.

(ii) Up to the action of $SO(n,1)$ there is a unique largest volume geodesic $n$-simplex. This turns out to be a regular ideal $n$-simplex. [See Munkholm]

## 2 The Boundary at Infinity of a Non-Positively Curved Symmetric Space

Rigidity is all about understanding how much fine structure actually exists implicitly at infinity.

### The Boundary at Infinity of a Metric Space

In the Poincaré disc model, $D^n$, of $\mathbb{H}^n$, there is an intuitive boundary at infinity, namely $S^{n-1}$. The following definition gives an intrinsic meaning to this.

**Definition 16.** Let $X$ be a metric space. The *boundary at infinity* of $X$, written $\partial X$, is defined to be the collection of asymptoty classes of geodesic rays $c : [0,\infty) \to X$. Given rays $c_1, c_2 : [0,\infty) \to X$ write $c_1(\infty) = c_2(\infty)$ if $c_1$ and $c_2$ are asymptotic, and hence represent the same element in $\partial X$.

**Lemma 17.** *Let $X$ be a complete* CAT(0) *space, and let $p \in X$. Then for every geodesic ray $c : [0,\infty) \to X$ there exists a unique geodesic ray $c' : [0,\infty) \to X$ asymptotic to $c$ such that $c'(0) = p$.*

It follows that in a complete CAT(0) space, and so in particular the class of symmetric spaces we are dealing with, we have the following alternative definition of the boundary at infinity:

**Definition 18.** Let $X$ be a complete CAT(0) space, and $X_0 \in X$ be a basepoint. Then $\partial X$ is the collection of geodesic rays $c : [0,\infty) \to X$ with $c(0) = X_0$.

## Topologising the Boundary

We want more structure on $\partial X$ than just that of a set. The following definition shows how to topologise $X \cup \partial X$ in the case where $X$ is a complete $CAT(0)$ space.

**Definition 19.** Let $X$ be a complete CAT(0) space and let $X_0$ be a basepoint. Given a geodesic ray $c : [0, \infty) \to X$ with $c(0) = X_0$ and positive numbers $R > 0$, $\epsilon > 0$ let

$$V_{\epsilon,R}(c) = \{c'(t) | c' : [0, \infty) \to X \,\text{is a geodesic ray}\,, c'(0) = X_0, t \in (R, \infty], d(c(R), c'(R)) < \epsilon\}.$$

Then the *cone topology* on $X \cup \partial X$ is the topology with basis the collection of open balls $B(x, r)$ for $x \in X$ together with the collection of all sets $V_{\epsilon,R}(c)$ for $c$ a geodesic ray with $c(0) = X_0$.

**Lemma 20.** *(i) The topology this gives on $X \cup \partial X$ is independent of the basepoint $X_0$.*

*(ii) Considering the Poincaré disc model for $\mathbb{H}^n$, the 'identity' map*

$$(\mathbb{H}^n \cup \partial\mathbb{H}^n, \text{Cone topology}) \to (\overline{B}(0, 1) \subseteq \mathbb{E}^n, \text{Euclidean topology})$$

*is a homeomorphism.*

## Metricising the Boundary

What more structure can we have at infinity? The following definition gives a first attempt at defining a metric on $\partial X$.

**Definition 21.** Let $X$ be a complete CAT(0) space and $X_0$ be a basepoint. Given $\xi_1, \xi_2 \in \partial X$ define $\angle_{X_0}(\xi_1, \xi_2)$ to be the angle between the unique geodesic rays $c_1, c_2$ with $c_i(0) = X_0$ and $c_i(\infty) = \xi_i$.

**Lemma 22.** *For any choice of basepoint $X_0$ the topology this metric induces on $\partial X$ agrees with the cone topology.*

But the metric depends on the choice of basepoint. To rectify this problem we make the following definition.

**Definition 23.** Let $X$ be a complete CAT(0) space. The *angular metric* on $\partial X$ is defined to be

$$\angle(\xi_1, \xi_2) = \sup_{X_0 \in X} \angle_{X_0}(\xi_1, \xi_2)$$

for $\xi_1, \xi_2 \in \partial X$.

**Examples:**

(i) In $\mathbb{E}^n$, $\angle_p(\xi_1, \xi_2) = \angle_q(\xi_1, \xi_2)$ for all $p, q \in \mathbb{E}^n$ and $\xi_1, \xi_2 \in \partial\mathbb{E}^n$. Thus $\angle = \angle_p$ for any choice of basepoint $p$. It follows that $\partial\mathbb{E}^n$ is isometric to $S^{n-1}$ via the map which takes $\xi \in \partial\mathbb{E}^n$ to the intersection of the geodesic ray from 0 to $\xi$ with the unit sphere centred at 0.

(ii) In $\mathbb{H}^n$ there is a unique geodesic joining any two distinct points $\xi_1, \xi_2 \in \partial\mathbb{H}^n$. Thus the angular metric is the discrete metric with $\angle(\xi_1, \xi_2) = \pi$ whenever $\xi_1 \neq \xi_2$.

## The Tits Boundary

We want to use the contrast in behaviour between examples (i) and (ii) to pick out Euclidean subspaces in an arbitrary symmetric space.

**Lemma 24.** *Let $X = \mathbb{H}^2 \times \mathbb{H}^2$. Then $(\partial X, \angle)$ is isometric to the spherical join $\mathbb{H}^2 * \mathbb{H}^2$, where each $\mathbb{H}^2$ has the discrete angular metric.*

Thus the metric picks out the flat directions. More generally if $X$ is a symmetric space of non-compact type then the boundary encodes the structure of the flats in $X$.

**Definition 25.** Let $X$ be a CAT(0) space. The *Tits metric* on $\partial X$ is the length metric $d_T$ associated to the angular metric $\angle$. The *Tits boundary* $\partial_T X$ of $X$ is $\partial_X$ together with this metric.

**Theorem 26** (Tits)**.** *Let $X$ be a higher rank irreducible symmetric space. Then $B = \partial_T X$ is a spherical Tits building and* Aut$(B)$*, the group of diagram automorphisms of the building, is 'exactly' the underlying algebraic group.*

For example, let $X = \frac{SL(n,\mathbb{R})}{O(n,\mathbb{R})}$ and $B = (\partial X, d_T)$. Then $\text{Aut}(X) \,'\!=\!'\, SL(n, \mathbb{R})$.

## The Visual Metric on $\partial\mathbb{H}^n$

Since the angular metric on $\partial\mathbb{H}^n$ is not very useful, we construct a better one.

**Definition 27.** Choose a basepoint $p \in \mathbb{H}^n$. Given $\xi_1, \xi_2 \in \partial\mathbb{H}^n$ define

$$\rho(\xi_1, \xi_2) = e^{-d(p,[\xi_1,\xi_2])}$$

where $[\xi_1, \xi_2]$ is the unique geodesic joining $\xi_1$ to $\xi_2$. The *visual metric* on $\partial\mathbb{H}^n$ is the pseudometric associated to $\rho$, which can be shown to be a genuine metric.

**Lemma 28.** *$\partial\mathbb{H}^n$ together with the visual metric is isometric to $S^{n-1}$ together with the usual round metric.*

However, it is all very well recovering the usual metric on $\partial\mathbb{H}^n \cong S^{n-1}$, but the action of Isom$(\mathbb{H}^n)$ extends continuously to an action on $\partial\mathbb{H}^n$ which does not preserve this metric. What does it preserve?

**Lemma 29.** *Let $\theta \in$ Isom$(\mathbb{H}^n)$ and let $\widetilde{\theta}$ be the induced map on $\partial\mathbb{H}^n$. Then $\widetilde{\theta}$ is conformal with respect to the visual (i.e. round) metric on $\partial\mathbb{H}^n$.*

*Proof.* Isom$(\mathbb{H}^n)$ is generated by reflections in co-dimension 1 hyperplanes, so it suffices to prove the result for such transformations. In the Poincaré ball model of $\mathbb{H}^n$ such a hyperplane is a sphere $S^{n-1}$ perpendicular to the boundary $\partial B(0, 1) \cong S^{n-1}$. Thus the induced map $\widetilde{\theta}$ is inversion in such a sphere and so is an element of Möb$(S^{n-1})$. Hence it is conformal. □

# 3 Sketch Proof of Mostow Rigidity

## Background Notions : Quasi-isometry

All proofs of Mostow Rigidity involve the notion of quasi-isometry.

**Definition 30.** A map $\varphi : (X_1, d_1) \to (X_2, d_2)$ between metric spaces is a quasi-isometry if there exist constants $\lambda \geq 1$, $\epsilon \geq 0$ and $C \geq 0$ such that

(i) for all $x, y \in X_1$ $\frac{1}{\lambda} d_1(x, y) - \epsilon \leq d_2(\varphi(x), \varphi(y)) \leq \lambda d_1(x, y) + \epsilon$ and

(ii) for all $z \in X_2$ there exists $x \in X_1$ such that $d_2(\varphi(x), z) < C$.

**Examples:**

(i) If $\Gamma_1, \Gamma_2$ are finitely generated groups and $\varphi : \Gamma_1 \to \Gamma_2$ is a homomorphism with finite kernel and finite index image then $\varphi$ is a quasi-isometry.

(ii) If a finitely generated group $\Gamma$ acts properly and co-compactly on a geodesic space $X$ then for any choice of basepoint $x_0 \in X$ the map $\Gamma \to X$ given by $\gamma \mapsto \gamma x_0$ is a quasi-isometry. Thus $\mathbb{Z}^2$ is quasi-isomatric to $\mathbb{E}^2$, and if $\Gamma$ is a co-compact subgroup of $SO(n, 1)$ then $\Gamma$ is quasi-isometric to $\mathbb{H}^n$. As a corollary we see that if $\Gamma_1, \Gamma_2$ are both co-compact lattices in $SO(n, 1)$ then $\Gamma_1$ is quasi-isometric to $\Gamma_2$.

## Proof in Rank 1 case

**Lemma 31** (Morse Lemma)**.** *Let $X = \mathbb{H}^n$, or more generally any $\delta$-hyperbolic space, and let $c : \mathbb{R} \to X$ be quasi-geodesic. Then there exists a unique geodesic $\gamma : \mathbb{R} \to X$ such that* $\mathrm{Im}(c) \subseteq N_\epsilon(\mathrm{Im}(\gamma))$ *for some $\epsilon$.*

This stability of quasi-geodesics allows us to consider $\partial X$ in terms of quasi-geodesic rays. Note that the equivalent statement is not true in general if $X$ is not $\delta$-hyperbolic. For example, in $\mathbb{E}^n$ consider the path $c : [0, \infty) \to \mathbb{E}^n$ given by $c(\theta) = r(\theta)e^{i\theta}$. If this rotates slowly enough, for example if $r(\theta) = \log(\theta)$, then this is a quasi-geodesic ray, but it is clearly not close to any geodesic ray.

*Sketch proof of rank 1 Mostow rigidity (theorem 2).* $\varphi : \Gamma_1 \to \Gamma_2$ is an isomorphism, so in particular is a quasi-isometry. We have two homomorphisms $\psi_1, \psi_2 : \Gamma_1 \to \mathrm{Aut}(\partial \mathbb{H}^n)$. $\psi_1$ is the obvious map

$$\Gamma_1 \hookrightarrow SO(n, 1) \to \mathrm{Conf}(\partial \mathbb{H}^n)$$

and $\psi_2$ is the map

$$\Gamma_1 \xrightarrow{\varphi} \Gamma_2 \hookrightarrow SO(n, 1) \to \mathrm{Conf}(\partial \mathbb{H}^n).$$

These two maps can be shown to be injective so we have two copies of $\Gamma_1$ in $\mathrm{Conf}(\partial \mathbb{H}^n)$. We want to construct $\gamma \in SO(n, 1) \subseteq \mathrm{Conf}(\partial \mathbb{H}^n)$ which conjugates these two copies. It can be shown that $\varphi$ induces a map $\partial\varphi \in \mathrm{QConf}(\partial \mathbb{H}^n)$, the group of quasi-conformal transformations of $\partial\mathbb{H}^n$ (which is roughly the collection of transformations which distort angles by a bounded amount), and that $\partial\varphi$ conjuagtes the two copies of $\Gamma_1$. Then use analysis to show that if subgroups are conjugate in $\mathrm{UQConf}(\partial\mathbb{H}^n)$, the group of uniform quasi-conformal transformations, then they are conjugate in $\mathrm{Conf}(\partial\mathbb{H}^n)$. □

## Proof in Higher Rank Case

NB: It is not true that every quasi-isometry $\mathbb{H}^n \to \mathbb{H}^n$ is Hausdorff close to an isometry.

The following result is the higher dimensional analogue of the Morse Lemma.

**Theorem 32** (Margulis, etc.). *The $X$ be a higher rank irreducible symmetric space. If $\varphi : X \to X$ is a quasi-isometry and $\mathbb{E} \hookrightarrow X$ is a maximal flat, then $\varphi(\mathbb{E})$ is in a bounded neighbourhood of a unique flat.*

**Corollary 33.** *Any quasi-isometry $\varphi : X \to X$ induces an automorphism of the the spherical Tits building $B = (\partial X, d_T)$.*

*Sketch proof of higher rank Mostow rigidity.* Given co-compact lattices $\Gamma_1, \Gamma_2$ and $\varphi : \Gamma_1 \to \Gamma_2$, we obtain as before $\psi_1, \psi_2 : \Gamma_1 \hookrightarrow \mathrm{Aut}(B)$. Then $\Gamma_1$ and $\Gamma_2$ are conjugate in $\mathrm{Aut}(B)$ by $\partial\varphi$. But $G = \mathrm{Aut}(B)$ is the original Lie group by Tits' theorem 23. □