Topics in Algebraic Surfaces:
(a lecture course at the SNS)

AC

Fall 2023

Abstract

This is a sketchy synopsis of the lectures according to my memory.

1 02 Oct

(1) Riemann–Roch and Serre duality; simple consequences: $\deg K = 2g - 2$;

(2) Curves of genus 1: models in $\mathbb{P}(1,2,3)$, $\mathbb{P}(1,1,2)$ and $\mathbb{P}^2$;

(3) Curves of genus 2, g^1_2 and hyperelliptic curves;

(4) The embedding lemma and the canonical embedding of nonhyperelliptic curves;

(5) Definition of special divisors.

Reference for this lecture: [2].

2 04 Oct

(1) Recap of last time: RR for curves (+ Serre duality) and (some) consequences, g^r_d, hyperelliptic curves, curves of genus 1, 2 and 3;

(2) Geometric Riemann–Roch. Corollary: general divisors are nonspecial. Nonsingular plane curves have genus $\frac{(d-1)(d-2)}{2}$. Canonical curves of genus 4: in general they have two g^1_3. A curve with a g^1_3 is called trigonal. Reference: [2];

(3) Canonical curves of genus 5: if a curve C of genus 5 has a g^1_3, then we can give a complete description of the (homogeneous) ideal of C in its canonical embedding. The curve lies on the scroll $\mathbb{F}(1,2)$. The equations of C in $\mathbb{P}^4$ with homogeneous coordinates $u_1, \ldots, u_5$ can be organized as

$$\operatorname{rk}\begin{pmatrix} u_1 & u_3 & u_4 \\ u_2 & u_4 & u_5 \end{pmatrix} < 2 \quad \text{and} \quad \begin{pmatrix} u_1 & u_3 & u_4 \\ u_2 & u_4 & u_5 \end{pmatrix} \cdot \begin{pmatrix} Q_3 \\ Q_2 \\ Q_1 \end{pmatrix} = 0$$

where Q_1, Q_2, and Q_3 are homogeneous quadratic polynomials in the variables $u_1, \ldots, u_5$; and then as anti-symmetric degeneracy locus [Folklore];

(4) Statement of the theorem of Buchsbaum–Eisenbud on codimension 3 Gorenstein rings [5, 4].

3 09 Oct

(1) Canonical curves of genus 6: statement of classification. A discussion of $\mathrm{Gr}(2,5)$ in its Plücker embedding; explicit description of a embedding of dP_5 as a linear section of $\mathrm{Gr}(2,5)$. [Folklore];

(2) Statement of the theorems of Max Noether and Enriques–Babbage–Petri [2];

(3) Short summary of Mukai–Ide classification of canonical curves of genus 8 and some speculations on (first) syzigies: If the homogeneous ideal I_C is generated by quadrics then perhaps the syzigies are linear unless there is a g^2_6?;

(4) The base point free pencil Lemma and idea of proof of M. Noether's — and hence also Petri's — Theorem [2];

(5) Why did I tell you all these things about canonical curves in a course on surfaces? (a) Because you know the Riemann–Roch theorem for curves but you don't know what to do with it; (b) This stuff on canonical curves will be useful when doing $K3$ surfaces.

4 11 Oct

(1) The intersection form on $\mathrm{Pic}(X)$ and its key properties (with indication of proof). Serre duality, Riemann–Roch, and Noether formula [3, Ch. I], [8, Ch. A];

(2) Example: intersection form on the scroll $\mathbb{F}_a$ [8, Ch. 2];

(3) Statement of: negativity of contractions and algebraic Hodge index theorem [8, A.7, D2.1, D2.2].

5 16 Oct

(1) Proof of negativity of contractions and algebraic Hodge index theorem [8];

(2) For X a projective variety over a field k, numerical and homological equivalence of divisors are the same modulo torsion. $\mathrm{NS}(X)$ is the group of divisors mod algebraic equivalence. $\mathrm{Pic}^\tau(X)$ is the group of divisors numerically equivalent to 0 mod algebraic equivalence: an algebraic group with connected component $\mathrm{Pic}^0(X)$. Hence there is $n > 0$ such that if D is num 0, then nD is alg 0, and hence homologically 0 for all reasonable cohomology theories;

(3) For X a proper (possibly singular) variety over a field k, the theorem of the base states that $\mathrm{NS}(X)$ is finitely generated. The only appropriate reference for this material is [1].

6 18 Oct

(1) Introduction to the numerical — and more generally discrete — invariants of surfaces and the relations that exist between them. Definition of birational maps; p_g and q are birational invariants (a very soft statement) [3], [8];

(2) A summary of what we get out of Hodge theory [6];

(3) If X is projective over $\mathbb{C}$, D a Cartier divisor, $L = O_X(D)$, then $c_1(L)$ (from the exponential sequence) is the same as $[D]$ (the cohomology class

of D) in $H^2(X;\mathbb{Z})$. This is because both classes are functorial under pullback and they coincide on $\mathbb{P}^n$. (I am not sure this matter is discussed properly anywhere.) [Folklore];

(4) Calculation of all these invariants for surfaces of degree d in $\mathbb{P}^3$;

(5) The theorem of Noether–Lefschetz, quartic surfaces that contain a line. The line is a -2curve and can be contracted to an ordinary double point on a 3, complete intersection in $\mathbb{P}(1^4,2)$. Unfortunately this is not the best example of what I am trying to demonstrate...

7 23 Oct

(1) A recap and continuation of the preceding discussion. If a quartic surface that contains a conic, then the conic can be contracted to a ODP on a $2,3$ complete intersection in $\mathbb{P}^4$. It follows that a quartic surface and a $2,3$ complete intersection in $\mathbb{P}^4$ are diffeomorphic (to see this I demonstrate the simultaneous resolution; it is a bit unfortunate because I haven't done examples of blow ups yet so this is a bit over the top). Similarly, a $2,3$ complete intersection in $\mathbb{P}^4$ containing a conic is diffeomorphic to a $2,2,2$ complete intersection in $\mathbb{P}^5$. André Weil definition of a $K3$ as a surface diffeomorphic to a quartic in $\mathbb{P}^3$.

(2) The theorem on resolution of indeterminacies via a sequence of blowing ups [3];

(3) First proof of the theorem on the factorization of birational morphisms by a strengthened version of the universal property of the blow up [3].

8 25 Oct

(1) Statement and proof of the Castelnuovo contractibility criterion for (-1)-curves on a surface;

(2) Second proof of the theorem on the factorization of birational morphisms. Method: starting from $f\colon Y \to X$, the canonical class K_Y is not nef over X, and then there is an exceptional curve E_i such that $K_Y \cdot E_i < 0$ and E_i is necessarily a (-1)-curve (let $C \subset X$ be reduced and irreducible; if $\chi(\mathcal{O}_C) \geq 1$, then C is a nonsingular $\mathbb{P}^1$ and $\chi(\mathcal{O}_C) = 1$.) [8, 4.15];

(3) An optimistic discussion of the rationality theorem and the minimal model program for surfaces. Motivation for the study of del Pezzo surfaces [8, D3.1, D4.1].

9 30 Oct

(1) A cubic surface has two disjoint lines and it is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in 5 points;

(2) Statement of the Kodaira vanishing theorem and consequences for a del Pezzo surface: for $q \geq 1$ $H^q(X, -K_X) = H^q(X, \mathcal{O})_X = (0)$. We can now use Riemann–Roch to compute $h^0(X, -K) = 1 + d$;

(3) Let C be a nonsingular curve of genus 1, L a line bundle on C of degree $d \geq 1$. Then we have a description of the graded ring $R = R(C, L)$:

   (i) If $d = 1$ then $R = k[x_1, x_2, y, z]/(z^2 + y^3 + yA_4 + A_6)$ (where $\deg z = 3$, $\deg y = 2$, $\deg x_i = 1$);

   (ii) If $d = 2$ then $R = k[x_1, x_2, x_3, y]/(y^2 + A_4)$ (where $\deg y = 2$, $\deg x_i = 1$);

   (iii) If $d = 1$ then R is generated in degree 1.

(4) The result is true in far greater generality for (possibly nonreduced) divisors $Z \subset X$ on a nonsingular surface but *much* harder to prove [7, Ch. 4];

(5) There is a corresponding description of the graded anticanonical ring $R(X, -K_X)$ of a del Pezzo surface. There are two extreme paths to prove it (and some in between): (a) pick and $Z \in |-K_X|$, prove the above for Z, and lift to X (very hard but useful for other purposes, e.g. study elliptic Gorenstein singularities of surfaces), and (b) show that $|-K_X|$ is base point free;

(6) Let X be a del Pezzo surface; then $|-K_X|$ has no fixed part. Method: write $A = F + M$; both A and M satisfy vanishing; compute with Riemann–Roch $h^0(M) < h^0(A)$;

(7) Let X be a del Pezzo surface of degree $d \geq 2$; then $|-K_X|$ has no fixed points. Method: assume that $x \in X$ is a fixed point; let $\pi\colon E \subset$

$Y \to x \in X$ be the blow up; then E is in the fixed part of $f^\star A$, that is $f^\star A = E + B$ where $h^0(Y, f^\star A) = H^0(Y, B)$. The key point is that: B is nef and big. Then again B satisfies vanishing and conclude with Riemann–Roch as before.

10 6 Nov

(1) Recap and wrap-up. End of the proof of classification of del Pezzo surfaces. The key point is this: assume a del Pezzo surface $X_d \subset \mathbb{P}^d$; let $x \in X$ be a point and consider the blow up (projection) $\pi\colon E \subset Y \to x \in X$. It is (pretty) clear that Y is a del Pezzo surface if and only if x does not lie on a line $L \subset X$. The set of lines on X is finite because it is a 0-dimensional (a line on X is a (-1)-curve hence it does not move

(2) Definition and first properties of $K3$ surfaces. Begin study of linear systems on $K3$ surfaces.

References

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