Course: M3/4/5P11 Galois Theory, Progress Test 2, 18/03/2020

This test is worth 20 marks. You are allowed to use any and all results proven in class. You don't need to state the results you use in full, a clear reference is sufficient.

Q 1 Consider field extensions $K \subset F_1, F_2 \subset L$. For each of the following statements, provide either a proof or a counterexample.

(a) If $K \subset F_1$ is separable, then $F_2 \subset F_1F_2$ is separable.

(b) If $F_2 \subset F_1F_2$ is normal, then $K \subset F_1$ is normal.

Q 2

(a) Draw a diagram showing all the intermediate fields of the extension $\mathbb{F}_3 \subset \mathbb{F}_{3^{12}}$ and the inclusions between them.

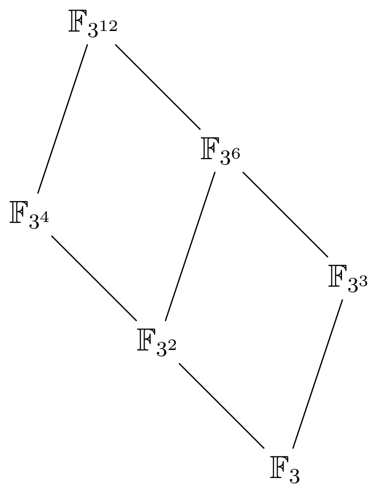
(b) How many monic degree 12 irreducible polynomials are there in $\mathbb{F}_3[X]$?

ANSWERS (Each part is worth **5 marks**)

A 1 (a) The statement is **true**. Indeed if $F_1 = K(a_1, \dots, a_r)$ then $F_1F_2 = F_2(a_1, \dots, a_r)$. To say that $K \subset F_1$ is separable is the same as saying that $a_1, \dots, a_r$ are separable over K. If this is the case, then a fortiori $a_1, \dots, a_r$ are separable over F_2, and then F_1F_2 is separable over F_2.

(b) The statement is **false**. A counterexample is given by the fields $K = \mathbb{Q}$, $F_1 = \mathbb{Q}(\sqrt[3]{2})$ (well-known to be non-normal), $F_2 = \mathbb{Q}(\sqrt{-3})$ (equivalently, $F_2 = \mathbb{Q}(\omega)$ where $\omega \in \mu_3$ is a primitive cube root of 1), and $F_1F_2 = L$ the splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 2$ and hence $\mathbb{Q} \subset F_1F_2$ is normal, and hence $F_2 \subset F_1F_2$ is also normal by general properties of normal extensions.

A 2 (a) This is the same as the diagram of divisors of 12. The question does not ask for a proof:



(b) In $\mathbb{F}_{3^{12}}$, we have $\mathbb{F}_{3^6} \cap \mathbb{F}_{3^4} = \mathbb{F}_9$. We count using the inclusion-exclusion principle:

$$x = \frac{3^{12} - 3^6 - 3^4 + 3^2}{12} = \frac{530,640}{12} = 44,220$$

(OF COURSE you didn't need to like, literally, calculate the actual number.)