Nairobi Examples

AC & LH

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1 Find $\int \frac{du}{5+4\sin u}$ by substituting $u = \tan\frac{u}{2}$. Do it again by setting

$$\sin u = \frac{e^{\mathtt{i}\,u} - e^{-\mathtt{i}\,u}}{2\,\mathtt{i}}$$

and reconcile the results you get.

2 Let $\Phi(x) \in \mathbb{C}(x)$ be a rational function.

(1) Show that there exist: a polynomial $P(x) \in \mathbb{C}[x]$, constants $A_i, a_i \in \mathbb{C}$ and integers $m_i \geq 1$ such that

$$\Phi(x) = P(x) + \sum_i \frac{A_i}{(x-a_i)^{m_i}}$$

(2) Use (1) to evaluate $\int_{x_0}^{x} \Phi(x)$. Note that it makes sense to either:

(a) consider $\Phi\colon I \to \mathbb{C}$ a function of a real variable in some interval $I = (c_1, c_2) \subset \mathbb{R}$, or

(b) $\Phi\colon \mathbb{C} \setminus \{a_i\} \to \mathbb{C}$ a function of a complex variable.

In the latter case the integral depends on the choice of a path in $\mathbb{C} \setminus \{a_i\}$ from x_0 to x and it is defined up to a constant

$$2\pi\mathtt{i} \sum_{m_i=1} k_i A_i \quad (k_i \in \mathbb{Z})$$

(3) Evaluate $\int \frac{x+1}{x^2+1}$ by the method in (2) above. Reconcile your result with the formula:

$$\int \frac{x+1}{x^2+1} dx = \frac{1}{2}\log|x^2+1| + \arctan x + C \quad (x \in \mathbb{R}).$$

3 Convince yourself that

$$\mathtt{i}\pi + 2\mathtt{i}\tan^{-1} x = \log \frac{x-i}{x+i}$$

4 Parametrize the conic $C = (x^2 + y^2 = 5)$ by considering a variable line through $(2, 1)$ and hence find all $x, y \in \mathbb{Q}$ such that $x^2 + y^2 = 5$.

5 Find the singular points of the following curves in $\mathbb{P}^2(\mathbb{C})$:

(a)

$$xz^2 - y^3 + xy^2 = 0\,;$$

(b)

$$(x + y + z)^3 - 27xyz = 0\,;$$

(c)

$$x^2y^2 + 36xz^3 + 24yz^3 + 108z^4 = 0\,.$$

6 For what values of t has the curve:

$$x^3 + y^3 + z^3 + t(x + y + z)^3 = 0$$

one or more singular points? Locate the singular points for each such value of t.

7 Find the intersections of the following pairs of curves in $\mathbb{P}^2$:

(a)

$$x(y^2 - xz)^2 - y^5 = 0$$
$$y^4 + y^3z - x^2z^2 = 0$$

(b)

$$x^3 - y^3 - 2xyz = 0$$
$$2x^3 - 4x^2y - 3xy^2 - y^3 - 2x^3z = 0$$

(c)

$$x^4 + y^4 - y^2z^2 = 0$$
$$x^4 + y^4 - 2y^3z - 2x^2yz - xy^2z + y^2z^2 = 0$$

8 Determine for what values $a, b \in \mathbb{C}$ the curve $(y^2 = x^3 + ax + b) \subset \mathbb{C}^2$ is nonsingular.

9 Show that an irreducible plane cubic with an ordinary node has three inflection points and that they are collinear.

10 Find a rational parametrization of the curves in $\mathbb{P}^2(\mathbb{C})$:

(a)

$$(x^2 + y^2)^2 = z^2(x^2 - y^2)$$

(b)

$$2x^4 - 3x^2yz + y^2z^2 - 2y^3z + y^4 = 0$$

11 Suppose given a rational parametrization of a plane curve:

$$x(t) = at^2 + bt + c$$
$$y(t) = pt^2 + qt + r$$

Find a plane algebraic curve $f(x, y) = 0$ such that $f\,(x(t), y(t)) = 0$.

12 Prove that for $u, v \in \mathbb{Z}$, $u^2 + v^2$ and $u^2 - v^2$ are both squares implies $v = 0$.

13 (1) Consider a polynomial differential operator $P = \sum_k P_k(D)t^k \in \mathbb{Z}[D, t]$ (where $D = t\,\frac{d}{dt}$). Show that a power series

$$f = \sum_n c_n t^n$$

is a solution of the ordinary differential equation $P \cdot f = 0$ if and only if its coefficients satisfy the *linear recursion relation*

$$\forall j \geq 0 : \sum_{k \leq j} P_k(j - k)\; c_{j-k} = 0.$$

(2) Show that for all integers $k \geq 0$ the power series

$$\widehat{I}_k(t) = \sum_{n=0}^{\infty} \frac{\big((8k+4)n\big)!n!}{(2n)!\big((2k+1)n\big)!^2\big((4k+1)n\big)!}\, t^n$$

is a solution of $H_k \cdot \widehat{I}_k = 0$ where $H_k = P_{k,0}(D) + tP_{k,1}(D)$, and

$$P_{k,0}(D) = -\prod_{i=0}^{2k}((2k+1)D - i)\prod_{i=0}^{4k}((4k+1)D - i) \quad \text{and}$$
$$P_{k,1}(D) = 4^{2k+1}(2k+1)\prod_{i\in\mathcal{I}_k}((8k+4)(D+1) - i),$$

where $\mathcal{I}_k = \{1, \dots 8k+3\} \setminus \{4k+2\} \cup \{2j\}_{j\in\{1,\dots 2k\}}$. Show the operator H_k is singular at 0, ∞ and at

$$t_k = \frac{(4k+1)^{4k+1}}{4^{8k+3}(2k+1)^{2k+1}}$$

14 For the Laurent polynomial

$$f(x,y) = x + y + \frac{1}{xy}$$

compute the power series expansion at $t = 0$ of the integral:

$$\pi_f(t) = \left(\frac{1}{2\pi\mathtt{i}}\right)^2 \oint \frac{1}{1 - tf(x,y)} \frac{dx}{x} \frac{dy}{y}$$

where you are integrating over the cycle $|x| = |y| = 1$. Show that $\pi_f(t)$ is a solution of the differential equation

$$\left[D^2 - 27t^3(D+1)(D+2)\right] \cdot \pi_f(t) = 0$$

References

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