Mirror symmetry and Fano orbifolds: an introduction*

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Abstract

This is an introduction to mirror symmetry for Fano orbifolds from a practical and, rather than theoretical, perspective. We assume very little prior knowledge of algebraic geometry.

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1 Introduction

1.1 What is mirror symmetry?

Mirror symmetry for Fano orbifolds. What is this? On the one hand we have X^n a Fano orbifold[1] of dimension n; on the other we have the mirror: a Landau–Ginzburg (LG) model $w : Y^n \to \mathbb{A}^1$, where w is a flat map with quasi-projective fibres.

One can say what Mirror Symmetry is at different "levels". The "high level" way is to state that $D^b_{coh} X \cong \mathrm{Fuk}(Y, w)$. The "low level" – and the approach we take here – is that there is an identity $\widehat{G}_X(t) = \pi_w(t)$. Here

$$G_X(t) = \sum_{d \geq 0} c_d t^d, \qquad \text{where } c_d = \int_{[X_{0,1,d}]^{\mathrm{vir}}} \psi^{d-2} \mathrm{ev}^*(\mathrm{pt}).$$

Here $X_{0,1,n}$ is the moduli stack of stable morphisms $\{f : (\Gamma, 0) \to X$ from a marked curve of genus 0 (Γ, x) to X, of degree d, that is $| \deg f^*(-K_X) = d\}$. Note also that the degree of $f^*(-K_X) > 0$ since X is Fano. The evaluation map is $\mathrm{ev} : X_{0,1,d} \to X, f \mapsto f(x)$, and $\psi = c_1(L)$ where L is the line bundle on $X_{0,1,d}$ with fibre $L_f = T^n_{\Gamma,x}$. Finally,

$$\widehat{G}_X(t) = \sum_{d \geq 0} d! c_d t^d, \qquad \text{and} \qquad \pi_w(t) = \int_\Gamma \frac{\Omega}{1 - wt},$$

where Ω is a holomorphic n-form on Y, normalised such that $\int_\Gamma \Omega = 1$, and Γ is some cycle in $H_n(Y; \mathbb{Z})$.

Example 1. Take $X = \mathbb{P}^2$. Then $Y = (\mathbb{C}^\times)^2$, $w = x + y + \frac{1}{xy}$, $\Gamma = S^1 \times S^1 = \{|x| = |y| = 1\}$, and $\Omega = \left(\frac{1}{2\pi i}\right)^2 \frac{dx}{x} \wedge \frac{dy}{y}$. Then

$$\pi_w(t) = \int_\Gamma \frac{\Omega}{1 - tw} = \sum t^d \int_\Gamma w^d \Omega = \sum c_0(w^d) t^d,$$

where the final equality comes from Cauchy's Theorem. Hence

$$\pi_w(t) = \sum_{d \geq 0} \binom{3d}{d, d, d} t^{3d}.$$

On the other side, Givental tells us the very nontrivial fact that

$$G_{\mathbb{P}^2} = \sum_{d \geq 0} \frac{t^{3d}}{(d!)^3}.$$

Hence $\widehat{G}_{\mathbb{P}^2} = \pi_w$.

These period functions $\pi_w(t)$ satisfy regular algebraic differential equations called *Picard–Fuchs equations*. So we can rephrase $\widehat{G}_X = \pi_w$ as:

regularised differential operator = Picard–Fuchs operator.

Write $L = \sum_{k=0}^r t^k P_k(D)$, where $D = t\frac{d}{dt}$. Let $\varphi = \sum c_n t^n$. Then $L \cdot \varphi \equiv 0$ if anf only if $\sum c_{n-k} P_k(n-k) = 0$.

Example 2. For $\mathbb{P}^2$, $L = D^2 - 27t^3(D + 1)(D + 2)$.

1.2 Some general open questions

Q1. How general is this picture? Do all Fano orbifolds have a mirror?

Q2. How precise is it? Can we define two sets $\mathcal{F}$ and $\mathcal{P}$ and a 1-to-1 function $\mathcal{F} \to \mathcal{P}$?

Q3. Can we make a directory of Fano/LG pairs?

We are interested in explicit constructions of Mirror Symmetry, and Q3 is our focus. One explicit construction is the Hori-Vafa construction.

[1] Here *orbifold* means that X is locally analytically $\mathbb{C}^n/G$, where G is finite; and*Fano* means that $\bigwedge^n T_X = -K_X$ is ample.

1.3 Toric complete intersections and the Hori–Vafa construction

What is a toric variety F? F is a quotient $[\mathbb{C}^m /\!\!/_\chi \mathbb{T}^r]$, where $\mathbb{T}^r = \operatorname{Spec} \mathbb{C}[\mathbb{L}]$ is an algebraic torus, $\mathbb{L} \cong \mathbb{Z}^r$ is a lattice of rank r, and where $\mathbb{T}^r$ acts on $\mathbb{C}^m$ via a linear representation.

We need to define the data for F and $X \subset F$:

$$D : \mathbb{Z}^m \to \mathbb{L}, \qquad \text{and line bundles } L_1, \dots, L_c \in \mathbb{L} \text{ on } F.$$

A toric complete intersection in F is $X = V(f_1) \cap \dots V(f_c)$, where $f_i \in \Gamma(F, L_i)$.

Let $X \subset F$ be a Fano toric complete intersection, defined via the data $(\mathbb{L}; D : \mathbb{Z}^n \to \mathbb{L}; L_1, \dots, l_c \in \mathbb{L})$, where $\mathbb{L} = \operatorname{Hom}(\mathbb{T}^r, \mathbb{C}^\times)$, $D : \mathbb{T}^\times \to (\mathbb{C}^\times)^m$, $F = [\mathbb{C}^m /\!\!/_\chi \mathbb{T}^r]$. The L_i give line bundles on F (since L_i is a character of $\mathbb{T}^r$). $\Gamma(F, L_i) = \{f \in \mathbb{C}[x_1, \dots, x_m] \mid g \in \mathbb{T}^r, f(g \cdot a) = L_i(g) f(a)\}$, $X = V(f_1) \cap \dots \cap V(f_c) \subset F$ where $f_i \in \Gamma(F, L_i)$ are general sections.

We probably want to put conditions on the data that ensure X is a Fano orbifold (quasi-smooth, well-formed).

Note that $D_i := D(e_i)$, $\{e_1, \dots, e_n\}$ a basis of $\mathbb{Z}^n$, gives a cone $\langle D_1, \dots, D_n \rangle = \sum \mathbb{R}_+[D_i] \subset \mathbb{L} \otimes \mathbb{R}$ which we require to be strongly convex (so that F is projective).

The key condition is that $K_X = (K_F + \sum L_i)\,|_X$, so we need that $-K_F - \sum L_i = \sum D_i - \sum L_i \in \operatorname{Amp} F$ for X to be Fano.

Key assumptions. For all $k \in [c] := \{1, \dots, c\}$ there exists $S_k \subset [m]$ such that $S_k \cap S_l = \emptyset$ and $L_k = \sum_{i \in S_k} D_i$. Choose a basis of $\mathbb{L}$. Write $D = (D_i^j)$. Define

$$Y := \left\{ \begin{array}{c|c} \prod x_i^{D_i^j} = 1 & j \in [r] \\ \sum_{i \in S_k} x_i = 1 & k \in [c] \end{array} \right\}$$

Define $w : Y \to \mathbb{C}$ by $\sum_{i \notin \bigcup S_j} x_i$. Then w is the *Hori–Vafa mirror.*

Remark 3.

- The "mirror theorem" should say that $\widehat{G}_X(t) = \pi_{Y,w}(t)$ under some conditions.
- We know this in some cases (e.g. F a manifold and L_i nef (Givental)) but not in the desired level of generality. In particular, we want the case when X is an orbifold.

Example 4. The cubic surface $X_3 \subset \mathbb{P}^3$ has "data" $D : \mathbb{Z}^4 \to \mathbb{Z} = \mathbb{L}$, $L = 3 \in \mathbb{Z}$, $D = (1\ 1\ 1\ 1)$ (corresponding to x_0, x_1, x_2, x_3). Givental tells us to:

$$I_X(t) = \sum_{d \geq 0} \frac{(3d)!}{(d!)^4} t^d = 1 + 6t + \dots$$

Thus we have:

$$G_X(t) = e^{-6t} I_X(t)$$

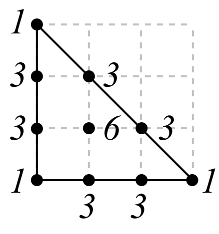
We pick Hori–Vafa data is $S = \{1, 2, 3\}$, $Y = \{x_0 x_1 x_2 x_3 = 1, x_1 + x_2 + x_3 = 1\}$, $w = x_0 : Y \to \mathbb{C}$. Want to write $Y = (\mathbb{C}^\times)^2$ and w a Laurent polynomial. To so this, solve for x_0:

$$Y = \{x_1 + x_2 + x_3 = 1\} \subset (\mathbb{C}^\times)^3 \hookleftarrow (\mathbb{C}^\times)^2_{x,y}$$

$$w = \frac{1}{x_1 x_2 x_3}$$

$$x_1 = \frac{1}{1 + x + y}, \qquad x_2 = \frac{x}{1 + x + y}, \qquad x_3 = \frac{y}{1 + x + y}$$

Hence $Y \cong (\mathbb{C}^\times)^2$ and $w = \frac{(1+x+y)^3}{xy}$, i.e.



$\subset \mathbb{Z}^2 = \{\text{monomials in } x, y\}$.

In this case

$$\pi_w(t) = \int \frac{\Omega}{1 - tw} = \widehat{I}_X(t)$$

and both satisfy the differential operator $D^2 - 3t(3D+1)(3D+2)$. Setting $w' = w - 6$ we have $\pi_{w'}(t) = \widehat{G}_X(t)$.

Example 5. Consider the toric complete intersection $X_{6,6} \subset \mathbb{P}(2,2,3,3,3)$. This is a nice orbifold with $4 \times \frac{1}{3}(1,1)$, $-K_X = oo(1)$, $(-K_X)^2 = 1/3$. It has

$$I_X(t) = \sum_{d \geq 0} \frac{(6d)!}{(2d)!^2(3d)!^3} t^d = 1 + 600t + \dots \qquad \Rightarrow G_X(t) = e^{-600t} I_X(t).$$

This has to do with the orbifold Gromov–Witten theory of X. But it does *not* satisfy the conditions of the Hori–Vafa construction, and does *not* have a toric degeneration (since $|-K_X| = \emptyset$). Even so, $\widehat{I}_X$ and $\widehat{G}_X$ are of geometric origin.

1.4 What we want to do

Definition 6. A *Fano polytope* is a lattice polytope $P \subset N \otimes \mathbb{R}$ such that

1. $0 \in \mathrm{int}(P)$;
2. the vertices of P are primitive vectors in $\mathbb{Z}^n = N$.

We have an equivalence $\{\text{Fano polytopes } P\} = \{\text{toric Fano variety } X_P\}$, where X_P is not necessarily smooth. Here X_P is the toric variety with fan given by the *spanning fan* of P.

Definition 7. A Fano orbifold X is class TG (*toric generisation*) if there exists a qG-degeneration of X to a toric Fano X_P.

Our view of mirror symmetry always starts from a Fano polytope P. Ideally we would have 1-1 correspondences

$$\begin{aligned} &\{\text{qG-families of class TG}\} \\ &\qquad \leftrightarrow \{\text{certain Fano polytopes } P\}/\text{mutation} \\ &\qquad \leftrightarrow \{\text{certain Laurent polynomials } f \text{ with } \mathrm{Newt}(f) = P\}/\text{mutation} \end{aligned}$$

Initial questionsr

Q1. Given P, understand the deformations of X_P to orbifolds.

Q2. Given P_1, P_2, when do X_{P_1}, X_{P_2} have deformations in common?

1.5 Exercises

(1) Consider the good-old complete elliptic integral

$$\pi(\lambda) = \frac{1}{2\pi i} \oint \frac{d\lambda}{\sqrt{x(x-1)(x-\lambda)}}$$

(i) By using the residue theorem around $\lambda = 0$, show the power series expansion

$$\pi(\lambda) = \sum_{n \geq 0} \binom{-1/2}{n}^2 \lambda^n$$

(ii) Conclude that $\pi(\lambda)$ is a solution of the Picard–Fuchs ordinary differential equation:

$$\Big(\lambda(\lambda-1)\frac{d^2}{d\lambda^2} + (2\lambda - 1)\frac{d}{d\lambda} + \frac{1}{4}\Big)\pi(\lambda) = 0$$

[For an alternative approach using Griffiths' reduction of pole method, see Clemens *A scrapbook of complex curve theory.*]

(2) In this question we study the Hori–Vafa construction. For simplicity we restrict to the case of complete intersections in weighted projective spaces (the general case of toric complete intersections is much more complicated but not mathematically much deeper).

(i) For Y n-dimensional, $f\colon Y \to \mathbb{C}$, $\Omega \in H^0(Y, \Omega^n_Y)$ a regular n-form and $\Gamma \in H_n(Y, \mathbb{Z})$ a n-cycle, consider

$$\pi(t) = \oint \frac{1}{1-tf}\Omega$$

Write $Y_t = \{y \in Y \mid f(y) = 1/t\}$, Assuming that the Griffiths "tube" map $\tau\colon H_{n-1}(Y_t;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is surjective, show that

$$\pi(t) = \oint_{\gamma(t)} \Omega(t)$$

where $\gamma(t) \in H_{n-1}(Y_t;\mathbb{Z})$ and $\Omega(t)$ is a $(n-1)$-form on Y_t.

[*Hint: residue theorem.*]

(ii) The Hori–Vafa mirror for the weighted projective space $X = \mathbb{P}(a_0, \dots, a_n)$ is the manifold

$$Y := \{x_0^{a_0} \cdots x_n^{a_n} = 1\} \subset \mathbb{C}^{\times\, n+1}$$

together with the morphism $W = x_0 + \cdots + x_n\colon Y \to \mathbb{C}$. Verify this by showing that

$$\widehat{I}(t) = \oint_\Gamma \frac{1}{1-t(x_0+\cdots+x_n)} \frac{1}{1-x_0^{-a_0}\cdots x_n^{-a_n}}\Omega$$

where:

- $$\widehat{I}(t) = \sum_{d\geq 0} \binom{(a_0+\cdots+a_n)d}{a_0 d, \dots, a_n d} t^{(a_0+\cdots a_n)d}$$

  is the regularized I-function of X.
- $\Gamma = \{|x_0| = \cdots = |x_n| = \varepsilon\} \cong (S^1)^{n+1} \subset \mathbb{C}^{\times\, n+1}$ and $\Omega = \left(\frac{1}{2\pi \mathtt{i}}\right)^{n+1} \frac{dx_0 \wedge \cdots \wedge dx_n}{x_0 \cdots x_n}$.
- You should use (i) to express this integral on $\mathbb{C}^{\times\, n+1}$ as an integral on Y.

[*Hint. Expand as a power series in t and use the residue theorem $n+1$ times.*]

(iii) Let now $X = X_d \subset \mathbb{P}(a_0, \cdots, a_n)$ be a degree d hypersurface. Assume that $d = a_0 + \cdots + a_k$. The Hori–Vafa mirror of X is the manifold

$$Y := \left\{ \begin{array}{c} x_0^{a_0} \cdots x_n^{a_n} = 1 \\ x_0 + \cdots + x_k = 1 \end{array} \right\} \subset \mathbb{C}^{\times\, n+1}$$

together with the morphism $W = x_0 + \cdots + x_n\colon Y \to \mathbb{C}$. Verify this by showing that:

$$\widehat{I}(t) = \oint_\Gamma \frac{1}{\big(1-t(x_0+\cdots+x_n)\big)\big(1-x_0^{-a_0}\cdots x_n^{-a_n}\big)\big(1-(x_0+\cdots+x_k)\big)}\,\Omega$$

where

- $$\widehat{I}(t) = \sum_{d\geq 0} \binom{(a_0+\cdots+a_k)d}{a_0 d, \dots, a_k d} \binom{(a_{k+1}+\cdots+a_n)d}{a_{k+1} d, \dots, a_n d} t^{(a_{k+1}+\cdots+a_n)d}$$

  is the regularized I-function of X.
- Γ and Ω are as in (ii).
- You should again interpret the integral on $\mathbb{C}^{\times\, n+1}$ as an integral on Y.

Generalize to higher-codimension complete intersections.

(3) In this question we study the (multiplicative, aka Hadamard) convolution of power series.

(i) Let $G = \sum B_n t^n$, $H = \sum C_n t^n$ be power series with radii of convergence $r_b, r_c > 0$. The Hadamard product $F = G \star H$ is the power series:

$$F(t) = \sum A_n t^n \quad \text{where} \quad A_n = B_n C_n$$

(with radius of convergence $r_a > r_b r_c$). Prove that for $|t| < r_b r_c$:

$$F(t) = \frac{1}{2\pi\mathtt{i}} \oint_{|u|=\rho} G(t/u)H(u)\frac{du}{u}$$

where $|t|/r_b < \rho < r_c$.

(So for example $I = \exp \star \widehat{I}$ is the Fourier–Laplace transform.)

(ii) We say that F is a period if $F(t) = \oint \frac{1}{1-tf}\Omega$ for some $f\colon Y \to \mathbb{C}$ as in Ex. 5(i) above.

Prove that if G, H are periods then so is $F = G \star H$. Revisit Ex. 2(iii) in this light.

[*Hint. Suppose that* G, H *are periods of* $g\colon Y \to \mathbb{C}$ *and* $h\colon Z \to \mathbb{C}$. *Consider* $Y \times Z$ *with* $f(y,z) = g(y)h(z)$.]

(4) In this question we find a mirror for the complete intersection $X = X_{6,6} \subset \mathbb{P}(2,2,3,3,3)$. This is based on calculations of **Don Zagier** and we are grateful to him for explaining this material to us and allowing us to include it here.

You will recall from the lectures that X is a perfectly nice, quasismooth and well-formed orbifold Fano surface with $4 \times 1/3(1,1)$ singularities and degree $K^2 = 1/3$. Because $h^0(X, -K_X) = (0)$, X does not have a toric degeneration. Also the Hori–Vafa construction does not apply in this case to give a mirror for X.

Nonetheless, we show that

$$\widehat{I}(x) = \sum_{n \geq 0} \frac{(6n)!^2 n!}{(2n)!^2 (3n)!^3} x^n$$

is a period. Looking at the other surfaces with $k \times 1/3(1,1)$-singularities, we might guess that $\widehat{I}(x)$ is the period of a rational differential on a pencil of curves of genus $5 = 1 + k$. Instead we will see that it is the period of a rational differential on a pencil of hyperelliptic curves of genus 4.

It will be really interesting to investigate this further, and see if some version of homological mirror symmetry holds in this case.

(i) Show that $\widehat{I}(x) = \sum A_n x^n$ where $A_n = B_n C_n$ with

$$B_n = \binom{6n}{3n} \quad \text{and} \quad C_n = \frac{(6n)!n!}{(3n)!(2n)!^2} = 2^{4n}\binom{3n - 1/2}{2n}$$

(ii) Let $G(x) = \sum B_n x^n$ and $H(x) = \sum C_n x^n$. Show that

$$\frac{1}{\sqrt{1-4x}} = \sum D_n x^n \quad \text{where} \quad D_{3n} = B_n$$

and convince yourself that

$$H\Big(\frac{t/8}{(1+2t)^3}\Big) = \frac{1-t}{1-4t}\sqrt{\frac{1+2t}{1+t/2}}$$

(Don didn't tell us how he came up with this identity. We used computer algebra to expand both sides in series in t and saw that the first 50 coefficients are the same).

(iii) Now use Ex. 3(i) to compute:

$$\widehat{I}(x) = \frac{1}{2\pi\mathtt{i}} \oint \frac{H^{(}u^3)}{\sqrt{1-4x/u}} \frac{du}{u}$$

Make the substitution $u = \frac{s}{1+16s^2}$ and use the second identity in (ii) to express $\widehat{I}(x)$ as the integral of a rational differential on the curve C_x:

$$y^2 = s(1+4s^3)(1+16s^3)\big(s-4x(1+16s^3)\big)$$

(the curve is a covering of the s-line with 10 ramification points).

2 Del Pezzo surfaces, mutations

3 A short course in toric geometry (starting from the end)

4 How to embed a toric variety in a toric variety