Trieste Examples

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Abstract

This is a list of exercises to go with the course *Mirror Symmetry for Fano orbifolds* held at the ICTP 01–12 August 2016 within the "Advanced School and Workshop on Moduli Spaces, Mirror Symmetry and Enumerative Geometry."

This sheet will be updated, corrected and expanded in the next days and weeks. Please let us know if you find mistakes.

Contents

1 Orbi-curves

These questions are to help you getting used to orbifolds. An orbi-curve is a one-dimensional orbifold $\big(\mathfrak{C}, x_i(r_i)\big)$ obtained from a nonsingular curve C with marked points $x_i \in C$ by introducing charts Δ/μ_{r_i} at x_i (where Δ is a small analytic disc).

(1) (i) Persuade yourselves that the orbifold fundamental group of an orbi-curve $\big(\mathfrak{C}, x_i(r_i)\big)$ is

$$\pi_1^{\text{orb}}\mathfrak{C} = \pi_1\big(C \smallsetminus \{x_i\}\big)/\langle \gamma_i^{r_i}\rangle$$

where γ_i are small loops around the punctures.

(ii) Let $\mathfrak{C}$ be an orbi-curve and G a finite group. Show that to give a representable morphism (a morphism of orbifolds is representable if it is injective on all stabilizers) $\mathfrak{C} \to BG$ is equivalent to give a group homomorphism $\pi_1^{\mathrm{orb}}\mathfrak{C} \to G$ which sends each γ_i to an element of order r_i. The data is also equivalent to give a principal G-bundle on $\mathfrak{C}$, that is a space $\pi \colon G \curvearrowright E \to C$ (where C is the coarse moduli space of $\mathfrak{C}$) which is a principal G-bundle over $C \smallsetminus \{x_i\}$ and has inertia group μ_{r_i} above x_i.

(2) Show Riemann-Roch and Serre duality for an orbi-curve $\mathfrak{C}$. For example, if L is a line bundle, then we get representations of μ_{r_i} on the fibre L_{x_i} of L at x_i and a Riemann-Roch formula

$$\chi(\mathfrak{C}, L) = \deg L + 1 - g - \sum \frac{k_i}{r_i}$$

(3) Give a sensible definition of "orbifold" *topological Euler number* of an orbi-curve. State some properties of the topological Euler number. Let C be a smooth proper curve and G a finite group acting on C: calculate the orbifold topological Euler number of the orbifold $[C/G]$ in terms of vertices, edges and faces of a G-invariant cellular decomposition of C.

2 Picard–Fuchs differential equations

(4) Consider the good-old complete elliptic integral

$$\pi(\lambda) = \frac{1}{2\pi\, i} \oint \frac{d\,\lambda}{\sqrt{x(x-1)(x-\lambda)}}$$

(i) By using the residue theorem around $\lambda = 0$, show the power series expansion

$$\pi(\lambda) = \sum_{n \geq 0} \binom{-1/2}{n}^2 \lambda^n$$

(ii) Conclude that $\pi(\lambda)$ is a solution of the Picard–Fuchs ordinary differential equation:

$$\Big(\lambda(\lambda-1)\frac{d^2}{d\,\lambda^2} + (2\lambda - 1)\frac{d}{d\lambda} + \frac{1}{4}\Big)\pi(\lambda) = 0$$

[For an alternative approach using Griffiths' reduction of pole method, see Clemens *A scrapbook of complex curve theory.*]

(5) In this question we study the Hori-Vafa construction. For simplicity we restrict to the case of complete intersections in weighted projective spaces (the general case of toric complete intersections is more complicated but not mathematically deeper).

(i) For Y n-dimensional, $f\colon Y \to \mathbb{C}$, $\Omega \in H^0(Y, \Omega_Y^n)$ a regular n-form and $\Gamma \in H_n(Y, \mathbb{Z})$ a n-cycle, consider

$$\pi(t) = \oint \frac{1}{1 - tf}\Omega$$

Write $Y_t = \{y \in Y \mid f(y) = 1/t\}$, Assuming that the Griffiths "tube" map $\tau\colon H_{n-1}(Y_t; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$ is surjective, show that

$$\pi(t) = \oint_{\gamma(t)} \Omega(t)$$

where $\gamma(t) \in H_{n-1}(Y_t; \mathbb{Z})$ and $\Omega(t)$ is a $(n-1)$-form on Y_t.

[*Hint: residue theorem.*]

(ii) The Hori–Vafa mirror for the weighted projective space $X = \mathbb{P}(a_0, \ldots, a_n)$ is the manifold

$$Y := \{x_0^{a_0} \cdots x_n^{a_n} = 1\} \subset \mathbb{C}^{\times\, n+1}$$

together with the morphism $W = x_0 + \cdots + x_n \colon Y \to \mathbb{C}$. Verify this by showing that

$$\widehat{I}(t) = \oint_\Gamma \frac{1}{1 - t(x_0 + \cdots + x_n)} \frac{1}{1 - x_0^{-a_0} \cdots x_n^{-a_n}}\Omega$$

where:

- $$\widehat{I}(t) = \sum_{d \geq 0} \binom{(a_0 + \cdots + a_n)d}{a_0 d, \ldots, a_n d} t^{(a_0 + \cdots a_n)d}$$

  is the regularized I-function of X.
- $\Gamma = \{|x_0| = \cdots = |x_n| = \varepsilon\} \cong (S^1)^{n+1} \subset \mathbb{C}^{\times\, n+1}$ and $\Omega = \left(\frac{1}{2\pi \mathtt{i}}\right)^{n+1} \frac{dx_0 \wedge \cdots \wedge dx_n}{x_0 \cdots x_n}$.
- You should use (i) to express this integral on $\mathbb{C}^{\times\, n+1}$ as an integral on Y.

[*Hint. Expand as a power series in t and use the residue theorem $n+1$ times.*]

(iii) Let now $X = X_d \subset \mathbb{P}(a_0, \cdots, a_n)$ be a degree d hypersurface. Assume that $d = a_0 + \cdots + a_k$. The Hori–Vafa mirror of X is the manifold

$$Y := \left\{ \begin{array}{c} x_0^{a_0} \cdots x_n^{a_n} = 1 \\ x_0 + \cdots + x_k = 1 \end{array} \right\} \subset \mathbb{C}^{\times\, n+1}$$

together with the morphism $W = x_0 + \cdots + x_n \colon Y \to \mathbb{C}$. Verify this by showing that:

$$\widehat{I}(t) = \oint_\Gamma \frac{1}{\big(1 - t(x_0 + \cdots + x_n)\big)\big(1 - x_0^{-a_0} \cdots x_n^{-a_n}\big)\big(1 - (x_0 + \cdots + x_k)\big)}\,\Omega$$

where

- $$\widehat{I}(t) = \sum_{d \geq 0} \binom{(a_0 + \cdots + a_k)d}{a_0 d, \ldots, a_k d} \binom{(a_{k+1} + \cdots + a_n)d}{a_{k+1} d, \ldots, a_n d} t^{(a_{k+1}+\cdots+a_n)d}$$
  is the regularized I-function of X.
- Γ and Ω are as in (ii).
- You should again interpret the integral on $\mathbb{C}^{\times\, n+1}$ as an integral on Y.

Generalize to higher-codimension complete intersections.

(6) In this question we study the (multiplicative) convolution of power series.

(i) Let $G = \sum B_n t^n$, $H = \sum C_n t^n$ be power series with radii of convergence $r_b, r_c > 0$. The Hadamard product $F = G \star H$ is the power series:

$$F(t) = \sum A_n t^n \quad \text{where} \quad A_n = B_n C_n$$

(with radius of convergence $r_a > r_b r_c$). Prove that for $|t| < r_b r_c$:

$$F(t) = \frac{1}{2\pi \mathtt{i}} \oint_{|u|=\rho} G(t/u) H(u) \frac{du}{u}$$

where $|t|/r_b < \rho < r_c$.

(So for example $I = \exp \star \widehat{I}$ is the Fourier–Laplace transform.)

(ii) We say that F is a period if $F(t) = \oint \frac{1}{1-tf}\Omega$ for some $f \colon Y \to \mathbb{C}$ as in Ex. 5(i) above.

Prove that if G, H are periods then so is $F = G \star H$. Revisit Ex. 5(iii) in this light.

[*Hint. Suppose that G, H are periods of $g \colon Y \to \mathbb{C}$ and $h \colon Z \to \mathbb{C}$. Consider $Y \times Z$ with $f(y,z) = g(y)h(z)$.*]

(7) In this question we find a mirror for the complete intersection $X = X_{6,6} \subset \mathbb{P}(2,2,3,3,3)$. This is based on calculations of **Don Zagier** and we are grateful to him for explaining this material to us and allowing us to include it here.

You will recall from the lectures that X is a perfectly nice, quasismooth and well-formed orbifold Fano surface with $4 \times 1/3(1,1)$ singularities and degree $K^2 = 1/3$. Because $h^0(X, -K_X) = (0)$, X does not have a toric degeneration. Also the Hori–Vafa construction does not apply in this case to give a mirror for X.

Nonetheless, we show that

$$\widehat{I}(x) = \sum_{n \geq 0} \frac{(6n)!^2 n!}{(2n)!^2 (3n)!^3} x^n$$

is a period. Looking at the other surfaces with $k \times 1/3(1,1)$-singularities, we might guess that $\widehat{I}(x)$ is the period of a rational differential on a pencil of curves of genus

$5 = 1 + k$. Instead we will see that it is the period of a rational differential on a pencil of hyperelliptic curves of genus 4.

It will be really interesting to investigate this further, and see if some version of homological mirror symmetry holds in this case.

(i) Show that $\widehat{I}(x) = \sum A_n x^n$ where $A_n = B_n C_n$ with

$$B_n = \binom{6n}{3n} \quad \text{and} \quad C_n = \frac{(6n)!n!}{(3n)!(2n)!^2} = 2^{4n}\binom{3n-1/2}{2n}$$

(ii) Let $G(x) = \sum B_n x^n$ and $H(x) = \sum C_n x^n$. Show that

$$\frac{1}{\sqrt{1-4x}} = \sum D_n x^n \quad \text{where} \quad D_{3n} = B_n$$

and convince yourself that

$$H\Big(\frac{t/8}{(1+2t)^3}\Big) = \frac{1-t}{1-4t}\sqrt{\frac{1+2t}{1+t/2}}$$

(Don didn't tell us how he came up with this identity. We used computer algebra to expand both sides in series in t and saw that the first 50 coefficients are the same).

(iii) Now use Ex. 6(i) to compute:

$$\widehat{I}(x) = \frac{1}{2\pi \mathtt{i}} \oint \frac{H^{(}u^3)}{\sqrt{1-4x/u}}\frac{du}{u}$$

Make the substitution $u = \frac{s}{1+16s^2}$ and use the second identity in (ii) to express $\widehat{I}(x)$ as the integral of a rational differential on the curve C_x:

$$y^2 = s(1+4s^3)(1+16s^3)\big(s - 4x(1+16s^3)\big)$$

(the curve is a covering of the s-line with 10 ramification points).

(8) (i) Choose a few (say 5) polygons from Figure 1 (make sure you choose them with different values of n);

(ii) For each of these polygons P figure out all of the mutations out of P and all the Laurent polynomials $f(x,y)$ with Newt $f = P$ that allow the same mutations;

(iii) For these Laurent polynomials, compute the "generic" genus of (the completion and normalization of) the curve $f(x,y) = 0$;

(iv) Can you now guess the answer to the following question: If $P \subset N_{\mathbb{R}}$ is a Fano polygon and $f(x,y)$ a generic Laurent polynomial with Newt $f = P$ that allows all the mutations out of P, what is the genus of the completion–normalization of the curve $f(x,y) = 0$? Can you prove it?

3 Fano polygons, polytopes and mutations

(9) Find 16 reflexive Fano polygons. (DON'T ask google.) For each of these polygons P, find all mutations out of P. Find all Laurent polynomials f with $\mathrm{Newt}\, f = P$ that allow the same mutations.

(10) Find an example of an "immutable" Fano polygon.
[Try $\mathbb{P}(3, 5, 11)$. Now find another one.]

(11) Let $P \subset \mathbb{R}^2$ be a Fano polygon. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be the vertices of P in counterclockwise order, and write $\mathbf{v}_i - \mathbf{v}_{i-1} = m_i \mathbf{e}_i$ where $m_i \in \mathbb{N}$ and $\mathbf{e}_i$ is a *primitive* lattice vector. Show that the Minkowski factors of P correspond to integer linear combinations:

$$\sum w_i \mathbf{e}_i = 0, \quad \text{where} \quad 0 \leq w_i < m_i$$

Generalize to higher dimensional Minkowski decompositions.

(12) Let $P \subset \mathbb{R}^3$ be a Fano polytope in 3D. Show that there are finitely mutations out of P. Generalize to all dimensions.

(13) In this question you will learn how to encode lattice polygons by using continued fractions representing zero.

(i) For integers $a_i \geq 2$ define

$$[a_0, \dots, a_n] = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_n}}}}$$

The purpose of this part of the question is to make sense of the expression $[a_0, \dots, a_n]$ for $a_i \in \mathbb{Z}$.

Indeed write:

$$\begin{pmatrix} -q_{n-1} & q_n \\ -p_{n-1} & p_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & a_m \end{pmatrix}$$

and define $[a_0, \dots, a_n] = \frac{p_n}{q_n}$. Show that this equals the above definition when all $a_i \geq 2$.

(ii) Let P be a Fano polygon. Construct a subdivision of the spanning fan of P corresponding to the minimal resolution of the toric surface X_P. Starting from a vertex $v = e_0$ of P, let $e_0, e_1, \dots, e_n$ be the generators of the rays of the subdivision in counterclockwise order. Show that

$$\begin{pmatrix} e_i \\ e_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_i \end{pmatrix} \begin{pmatrix} e_{i-1} \\ e_i \end{pmatrix}$$

where $-a_i = E_i^2$ is the selfintersection of the corresponding divisor in the minimal resolution.

(iii) Show that $[a_0, ..., a_n] = 0$ "with multiplicity one". Viceversa, starting with a sequence $a_0, \dots, a_n$ of integers such that $[a_0, \dots, a_n] = 0$ with multiplicity one, construct a corresponding Fano polygon.

(14) Write down and understand the complete mutation graph of the polygon with vertex matrix:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

4 Toric complete intersections

(15) The entries in (i–iv) below are data for toric complete intersections: the matrix to the left of the vertical bar is interpreted as a linear homomorphism $D\colon \mathbb{Z}^m \to \mathbb{Z}^r = \mathbb{L}^*$ corresponding to a representation of $\mathbb{C}^{\times r} = \mathbb{T}^r$ on $\mathbb{C}^m$, and this in turn is seen as the datum for a toric orbifold $F = [\mathbb{C}^m /\!/ \mathbb{T}^r]$. To the right of the vertical bar are $c \geq 1$ column vectors in $\mathbb{Z}^r = \mathbb{L}^*$ that we interpret as characters $\chi_1, \dots, \chi_c$ of $\mathbb{T}^r$ and hence line bundles $L_1, \dots, L_c$ on F.

In this question we study the toric complete intersection

$$X := V(f_1, \dots, f_c)$$

where

$$f_i \in \Gamma(\chi_i) = \Big\{ f \in \mathbb{C}[x_1, \dots, x_m] \mid \forall\, a \in \mathbb{C}^m,\ \forall\, \lambda \in \mathbb{T}^r,\ f(\lambda a) = \chi_i(\lambda) f(a) \Big\}$$

is a general section of L_i. (In all cases X is a surface.)

In all cases:

- Fix a stability condition such that X is Fano;
- Find all singularities of X and thus show that X is quasi-smooth and well-formed;
- Compute the self-intersection number K_X^2.

(i)

$$\left(\begin{array}{cccccc|cc} 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 & 3 & 4 & 1 \end{array}\right)$$

(ii)

$$\left(\begin{array}{cccccc|cc} 1 & 1 & 2 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 \end{array}\right)$$

(iii)

$$\left(\begin{array}{cccccc|cc} 1 & 3 & 1 & 1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 & 1 & 1 & 2 & 4 \end{array}\right)$$

(iv)

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 3 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 & 3 & 0 & 3 \\ 3 & 0 & 1 & 3 & 3 & 0 & 6 \end{array}$$

5 Struts

(16) Choose five polygons from the Appendix; for each of them, find a scaffolding of P by struts; then write down the corresponding toric complete intersection data. Convince yourself by any means that X_P is given by the appropriate binomials as a complete intersection is this family. Verify that the generic member of the family is a well-formed and quasi-smooth orbifold with $1/3(1,1)$ singularities. Also, write out the corresponding mirror Laurent polynomial.

Do the same for a few 3-dimensional reflexive polytopes. (You can look at them in SAGE).

6 Appendix

Table 2: Complete intersection descriptions of the representatives of Table 1 for the 26 mutation-equivalence classes with singularity content $(n, \{m \times \frac{1}{3}(1,1)\})$.

#	Weights and line bundles
1	$\begin{array}{cccc\|c} 1 & 2 & 3 & 5 & 10 \end{array}$
2	$\begin{array}{ccccc\|cc} 1 & 3 & 3 & 2 & 2 & 6 & 4 \end{array}$
3	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 & 2 & 4 \end{array}$
4	$\begin{array}{ccccc\|c} 1 & 0 & 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 & 1 & 4 \end{array}$
5	$\begin{array}{cccccc\|cc} 1 & 0 & 2 & 2 & 1 & 1 & 4 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 & 4 \end{array}$
6	$\begin{array}{ccccc\|c} 1 & 0 & 1 & 1 & 2 & 4 \\ 1 & 1 & 0 & 3 & 3 & 6 \end{array}$

7	none
8	$\begin{array}{cccccc\|c} 1 & 0 & 1 & 0 & 3 & 2 & 5 \\ 0 & 1 & 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \end{array}$
9	none
10	none
11	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 3 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 & 3 & 0 & 3 \\ 2 & 0 & 1 & 3 & 3 & 0 & 6 \end{array}$
12	$\begin{array}{cccc\|c} 1 & 3 & 3 & 1 & 6 \end{array}$
13	$\begin{array}{ccccc\|c} 1 & 0 & 1 & 1 & 1 & 3 \\ 1 & 1 & 3 & 0 & 0 & 3 \end{array}$
14	$\begin{array}{ccccc\|c} 1 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 0 & 0 & 3 \end{array}$
15	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 1 & 3 & 4 \end{array}$
16	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 4 & 0 & 0 & 4 \end{array}$
17	$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{array}$
18	$\begin{array}{ccccc\|c} 1 & 1 & 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}$
19	$\begin{array}{cccccc\|c} 1 & 1 & 0 & 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 5 & 0 & 0 & 5 \end{array}$

20	$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 3 \end{array}$
21	$\left.\begin{array}{cccc} 1 & 1 & 3 & 1 \end{array}\right| 4$
22	$\begin{array}{ccccc\|c} 1 & 0 & 1 & 3 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{array}$
23	$\begin{array}{ccccc\|c} 1 & 0 & 3 & -1 & 1 & 2 \\ 2 & 1 & 3 & 1 & 0 & 4 \end{array}$
24	$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 3 \end{array}$
25	$\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}$
26	$\begin{array}{ccc} 1 & 1 & 3 \end{array}$

Table 1: Representatives for the 26 mutation-equivalence classes of Fano polygons $P \subset N_{\mathbb{Q}}$ with singularity content $(n, \{m \times \frac{1}{3}(1,1)\})$. The degrees $-K_X^2 = 12 - n - \frac{5m}{3}$ of the corresponding toric varieties are also given. See also Figure 1.

#	$\mathcal{V}(P)$	n	m	$-K_X^2$
1	$(7,5),(-3,5),(-3,-5)$	10	1	$\frac{1}{3}$
2	$(3,2),(-3,2),(-3,-2),(3,-2)$	8	2	$\frac{2}{3}$
3	$(3,1),(3,2),(-1,2),(-2,1),(-2,-3),(-1,-3)$	6	3	1
4	$(3,2),(-1,2),(-2,1),(-2,-3)$	9	1	$\frac{4}{3}$
5	$(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2),(1,-2),(2,-1)$	4	4	$\frac{4}{3}$
6	$(3,2),(-1,2),(-2,1),(-2,-1),(-1,-2)$	7	2	$\frac{5}{3}$
7	$(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2),(1,-1)$	2	5	$\frac{5}{3}$
8	$(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2)$	5	3	2
9	$(1,1),(-1,2),(-2,1),(-1,-1),(1,-2),(2,-1)$	0	6	2
10	$(1,1),(-1,2),(-1,-2),(1,-2)$	8	1	$\frac{7}{3}$
11	$(1,1),(-1,2),(-2,1),(-1,-1),(2,-1)$	3	4	$\frac{7}{3}$
12	$(3,1),(-3,1),(0,-1)$	6	2	$\frac{8}{3}$
13	$(1,1),(-1,2),(-1,-1),(2,-1)$	6	2	$\frac{8}{3}$
14	$(1,1),(-1,2),(-2,1),(-1,-1),(1,-1)$	4	3	3
15	$(1,1),(-1,2),(-1,-1),(1,-1)$	7	1	$\frac{10}{3}$
16	$(1,1),(-1,2),(-1,0),(0,-1),(2,-1)$	5	2	$\frac{11}{3}$
17	$(1,0),(1,1),(-1,2),(-2,1),(-1,-1),(0,-1)$	3	3	4
18	$(1,0),(0,1),(-1,1),(-1,-3)$	6	1	$\frac{13}{3}$
19	$(1,1),(-1,2),(-1,1),(0,-1),(2,-1)$	4	2	$\frac{14}{3}$
20	$(1,1),(-1,2),(-2,1),(-1,-1),(0,-1)$	2	3	5
21	$(1,1),(-1,2),(-1,-2)$	5	1	$\frac{16}{3}$
22	$(1,1),(-1,2),(-1,-1),(0,-1)$	5	1	$\frac{16}{3}$
23	$(1,1),(-1,2),(0,-1),(2,-1)$	3	2	$\frac{17}{3}$
24	$(0,1),(-1,2),(-2,1),(-1,0),(1,-1)$	4	1	$\frac{19}{3}$
25	$(0,1),(-1,2),(-2,1),(1,-1)$	3	1	$\frac{22}{3}$
26	$(-1,2),(-2,1),(1,-1)$	2	1	$\frac{25}{3}$

Figure 1: Representatives for the 26 mutation-equivalence classes of Fano polygons $P \subset N_{\mathbb{Q}}$ with singularity content $(n, \{m \times \frac{1}{3}(1,1)\})$. See also Table 1.