Udine Examples

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Abstract

This is a list of exercises to go with the EMS School “New perspectives on the classification of Fano manifolds” in Udine, Sep 29–Oct 03, 2014. [1]

Contents

General advice Realistically, there is no way you can do all of these problems during the School. You should choose some that you like and get help setting yourself up. Help is available by the following people on the following topics:

Mohammad Akhtar Mutations;

Liana Heuberger Classification of the 29 (26) surfaces, toric complete intersections;

Alessandro Oneto Quantum orbifold cohomology;

Andrea Petracci Quantum orbifold cohomology;

[1] This sheet was put together in great haste and I expect that it contains several errors and misprints. I keep a corrected and updated version of this document on my teaching page `http://wwwf.imperial.ac.uk/ acorti/teaching.html`

Thomas Prince Mutations, toric complete intersections; Picard–Fuchs operators;

Ketil Tveiten Rigid maximally mutable Laurent polynomials. Topology of $f(x, y) = 0$. Monodromy and ramification index.

1 Log del Pezzo surfaces

(1) Let $\mathfrak{X}$ be the stack $[(xy + z^w = 0)/\mu_r]$ where μ_r acts with weights $(1, wa - 1, a)$ and $\mathrm{hcf}(r, a) = 1$ and $\mathrm{hcf}(wr, wa-1) = 1$ (so the moduli space X is the surface quotient singularity $1/n(1, p-1)$ with $n = wr$, $p = wa$). Show that $\mathrm{Ext}^1_{\mathcal{O}_{\mathfrak{X}}}(\Omega^1_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \cong \mathbb{C}^{m-1}$ where $w = mr + w_0$, $0 \le w_0 < r$.

(2) Let the isolated surface quotient singularity $X = 1/n(1, p-1)$ have Gorenstein index $r = 3$. Show that: either X is of class T; or $X = \frac{1}{3(3m+2)}(1, 3m+1)$ with content $\left\{m, \frac{1}{6}(1,1)\right\}$; or $X = \frac{1}{3(3m+1)}(1, 6m+1)$ with content $\left\{m, \frac{1}{3}(1,1)\right\}$. Draw some of the cones and stare at them.

Do a similar analysis for Gorenstein index 2, 4 and 5.

(3) Consider a del Pezzo surface X with k quotient singularities $1/3(1, 1)$.

(i) Somehow or other convince yourself that

$$h^0(X, -nK) = 1 + \frac{n(n+1)}{2}K^2 + k \times \begin{cases} -\frac{1}{3} \text{ if } n \equiv 1 \pmod 3 \\ 0 \text{ if } n \equiv 0, 2 \pmod 3 \end{cases}$$

(the official reference of this kind of thing is Reid's *Young Person's Guide.*)

(ii) Show that $K_X^2 = 12 - n - \frac{5k}{3}$ where $n \in \mathbb{Z}$ is the topological Euler number $e(X^0)$ of the smooth locus of X, i.e. the complement $X \setminus S$ of the set S of singular points of X.

(iii) Assume that $k = 1$ and X has no free (-1)-curves (that is, (-1)-curves contained in the smooth locus of X). Show that $X = \mathbb{P}(1, 1, 3)$. (Hint: use the minimal model program.)

(iv) Conclude that if $k = 1$ then X is the blow up of $\mathbb{P}(1, 1, 3)$ at ≤ 8 general points.

(v) (Harder) Study the case $k = 2$ in a similar vein.

(4) Procure yourself 5 general lines in $\mathbb{P}^2$, blow up the 10 points of intersection to obtain a surface Y, contract the strict transforms of the 5 general lines to obtain a surface X. Show that X is a Del Pezzo surface with 5 singular points $1/3(1, 1)$ and degree $K^2 = 2/3$. Show that X has a free (-1)-curve, that is a (-1)-curve not intersecting any of the singular points. Contracting this (-1)-curve yields a surface X_1 with 5 singular points $1/3(1, 1)$ and degree $K_X^2 = 2/3$. Show that X_1 in turn has no free (-1)-curves. Give an alternative Italian-style birational geometry construction of X_1. (I don't know how to do this.)

2 Fano polygons, polytopes and mutations

(5) Find 16 reflexive Fano polygons. (DON'T ask google) and find all the maximally mutable Laurent polynomials supported on them.

(6) Find an example of an "immutable" Fano polygon.

[Try $\mathbb{P}(3,5,11)$. Now find another one.]

(7) A polygon P is called centrally symmetric if $v \in P$ implies that $-v \in P$. Show that any centrally symmetric polygon is minimal. It is conjectured that there exists at most one centrally symmetric polygon in each mutation equivalence class – can you prove this?

(8) Let $P \subset \mathbb{R}^2$ be a Fano polygon. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the vertices of P in counter-clockwise order, and write $\mathbf{v}_i - \mathbf{v}_{i-1} = m_i \mathbf{e}_i$ where $m_i \in \mathbb{N}$ and $\mathbf{e}_i$ is a *primitive* lattice vector. Show that the Minkowski factors of P correspond to integer linear combinations:

$$\sum w_i \mathbf{e}_i = 0, \quad \text{where} \quad 0 \leq w_i < m_i$$

Generalize to higher dimensional Minkowski decompositions.

(9) Let $P \subset \mathbb{R}^3$ be a Fano polytope in 3D. Show that there are finitely mutations out of P.

(10) In this question you will learn how to encode lattice polygons by using continued fractions representing zero.

(i) For integers $a_i \geq 2$ define

$$[a_0, \ldots, a_n] = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_n}}}}$$

The purpose of this part of the question is to make sense of the expression $[a_0, \ldots, a_n]$ for $a_i \in \mathbb{Z}$.

Indeed write:

$$\begin{pmatrix} -q_{n-1} & q_n \\ -p_{n-1} & p_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & a_m \end{pmatrix}$$

and define $[a_0, \ldots, a_n] = \frac{p_n}{q_n}$. Show that this equals the above definition when all $a_i \geq 2$.

(ii) Let P be a Fano polygon. Construct a subdivision of the spanning fan of P corresponding to the minimal resolution of the toric surface X_P. Starting from a vertex $v = e_0$ of P, let $e_0, e_1, \dots, e_n$ be the generators of the rays of the subdivision in counterclockwise order. Show that

$$\begin{pmatrix} e_i \\ e_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_i \end{pmatrix} \begin{pmatrix} e_{i-1} \\ e_i \end{pmatrix}$$

where $-a_i = E_i^2$ is the selfintersection of the corresponding divisor in the minimal resolution.

(iii) Show that $[a_0, ..., a_n] = 0$ "with multiplicity one". Viceversa, starting with a sequence $a_0, \dots, a_n$ of integers such that $[a_0, \dots, a_n] = 0$ with multiplicity one, construct a corresponding Fano polygon.

(11) Write down and understand the complete mutation graph of the polygon with vertex matrix:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

(we don't know how to do this. If you do, let us know.)

3 Toric complete intersections

(12) Find all singularities of X and compute K_X^2 for X one of the following toric complete intersections:

(i)

$$\begin{array}{cccccc|cc} 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 & 3 & 4 & 1 \end{array}$$

(ii)

$$\begin{array}{cccccc|cc} 1 & 1 & 2 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 \end{array}$$

(iii)

$$\begin{array}{cccccc|cc} 1 & 3 & 1 & 1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 & 1 & 1 & 2 & 4 \end{array}$$

(iv)

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 3 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 & 3 & 0 & 3 \\ 3 & 0 & 1 & 3 & 3 & 0 & 6 \end{array}$$

4 Picard–Fuchs differential equations

(13) Consider the good-old complete elliptic integral

$$\pi(\lambda) = \frac{1}{2\pi\, i} \oint \frac{d\,\lambda}{\sqrt{x(x-1)(x-\lambda)}}$$

(i) By using the residue theorem around $\lambda = 0$, show the power series expansion

$$\pi(\lambda) = \sum_{n\geq 0} \binom{-1/2}{n}^2 \lambda^n$$

(ii) Conclude that $\pi(\lambda)$ is a solution of the Picard–Fuchs ordinary differential equation:

$$\Big(\lambda(\lambda-1)\frac{d^2}{d\,\lambda^2} + (2\lambda - 1)\frac{d}{d\lambda} + \frac{1}{4}\Big)\pi(\lambda) = 0$$

[For an alternative approach see Clemens *A scrapbook of complex curve theory.*]

(14) (i) Choose a few (say 5) polygons from Figure 4 (make sure you choose them with different values of n);

(ii) For each of these polygons P figure out all of the maximally mutable Laurent polynomials $f(x, y)$ with $\operatorname{Newt} f = P$;

(iii) Compute the "generic" genus of the completion and normalization of the curve $f(x, y) = 0$;

(iv) Can you now guess the answer to the following question: If $P \subset N_{\mathbb{R}}$ is a Fano polygon and $f(x, y)$ a generic maximally mutable Laurent polynomial with $\operatorname{Newt} f = P$, what is the genus of the completion–normalization of the curve $f(x, y) = 0$? Can you prove it?

5 Quantum cohomology

In questions 15 and 16, you are asked to compute part of the small quantum cohomology of Del Pezzo surfaces of degree 2 and 3 naïvely from the definition.

(15) Cubic surface (i) Let $X = X_3^2 \subset \mathbb{P}^3$ be a nonsingular cubic surface. Let A be the class of a hyperplane section and consider the subspace of $H^\bullet X$ with basis $\mathbf{1}, A, A^2 = 3\text{pt}$. Show that quantum multiplication by A preserves this subspace and it is given by the matrix

$$M = \begin{pmatrix} 0 & 108q^2 & 756q^3 \\ 1 & 9q & 108q^2 \\ 0 & 1 & 0 \end{pmatrix}$$

[Hint. The relevant enumerative information is: $\langle A, A, A\rangle_1 = 27$, $\langle A, A, A^2\rangle_2 = 12\langle \text{pt}\rangle_2 = 12 \times 27$, $\langle A, A^2, A^2\rangle_3 = 27 \times \langle \text{pt}, \text{pt}\rangle_3 = 27 \times 84$.]

(ii) This is one of the simples examples of a mirror theorem: $e^{6q}\psi_0$—cf. Q 8(4)—satisfies the hypergeometric operator

$$D^3 - 3q(3D + 1)(3D + 2).$$

(16) Degree two del Pezzo $X = X_4^2 \subset \mathbb{P}(1^3, 2)$ Make a similar discussion for the del Pezzo surface of degree 2, $X = X_4^2 \subset \mathbb{P}(1^3, 2)$:

(i) In the obvious basis

$$M = \begin{pmatrix} 0 & 552q^2 & 7,488q^3 \\ 1 & 28q & 552q^2 \\ 0 & 1 & 0 \end{pmatrix}$$

(For instance, $7,488 = 6 \times 1248$, where 1248 is the number of cubics through two general points of X.)

(ii) Make contact with the appropriate hypergeometric differential operator $D^3 - 4q(4D+1)(4D+3)$.

The next 7 questions form a series on the basics of *orbi-curves*, which should help to later build a feeling for how quantum orbifold cohomology is put together. An orbi-curve is a nodal twisted curve $\big(\mathfrak{C}, x_i(r_i)\big)$ where all points with non-trivial stabiliser are marked with an isomorphism $G_{x_i} = \mu_{r_i}$ (sometimes I omit the marked points from the notation).

(17) (i) Persuade yourselves that the orbifold fundamental group of a smooth orbi-curve $\big(\mathfrak{C}, x_i(r_i)\big)$ is

$$\pi_1^{\mathrm{orb}}\mathfrak{C} = \pi_1\big(\mathfrak{C} \smallsetminus \{x_i\}\big)/\langle \gamma_i^{r_i} \rangle$$

where γ_i are small loops around the punctures.

(ii) Let $\mathfrak{C}$ be a smooth orbi-curve and G a finite group. Show that to give a representable morphism $\mathfrak{C} \to BG$ is equivalent to give a group homomorphism $\pi_1^{\mathrm{orb}}\mathfrak{C} \to G$ which sends each γ_i to an element of order r_i. The data is also equivalent to give a principal G-bundle on $\mathfrak{C}$, that is a space $\pi\colon G \curvearrowright E \to C$ (where C is the coarse moduli space of $\mathfrak{C}$) which is a principal G-bundle over $C \smallsetminus \{x_i\}$ and has inertia group μ_{r_i} above x_i.

(18) Show Riemann-Roch and Serre duality for an orbi-curve $\mathfrak{C}$. For example, if L is a line bundle, then we get representations of μ_{r_i} on the fibre L_{x_i} of L at x_i and a Riemann-Roch formula

$$\chi(\mathfrak{C}, L) = \deg L + 1 - g - \sum \frac{k_i}{r_i}$$

(19) If $\big(\mathfrak{C}; x_i(r_i)\big)$ is a n-pointed orbi-curve and $f\colon \big(\mathfrak{C}; x_i(r_i)\big) \to \mathfrak{X}$ is a stable representable morphism, then $f^*T_{\mathfrak{X}}$ makes sense is an orbi-bundle and $\mu_{r_i} = G_{x_i}$ acts through the representation into $G_{f(x)}$; we label the representation at x_i by its weights

$0 \le w_{i,j} < r_i$; persuade yourself that the expected dimension of the moduli space is

$$\dim \mathfrak{X}_{0,n,\beta} = \chi\big(\mathfrak{C}, f^* T_{\mathfrak{X}}\big) + n - 3 =$$
$$= -K_{\mathfrak{X}} \cdot \beta + \dim \mathfrak{X} + n - 3 - \sum_{i=1}^{n} \sum_{j=1}^{\dim \mathcal{X}} \frac{w_{i,j}}{r_i}$$

[Hint. Denote by

$$\mathbb{L}_f^\bullet = f^*\Omega^1_{\mathfrak{X}} \to \Omega^1_{\mathfrak{C}}$$

the cotangent complex of the morphism f. The deformation theory of f is controlled by the hyperext algebra $\operatorname{Ext}^\bullet\big(\mathbb{L}_f, \mathcal{O}_{\mathfrak{C}}\big)$.]

(20) Let $f\colon \big(\mathfrak{C}, x_i(r_i)\big) \to \mathfrak{X}$ be a stable morphism. Let us assume that f is a embedding locally at every point of $\mathfrak{C}$. In this case, there is a locally free sheaf N_f, the *normal bundle* of f, defined by the sequence

$$0 \to N_f^\vee \to \Omega^1_{\mathfrak{X}} \to \Omega^1_{\mathfrak{C}} \to 0$$

This needs to be taken with a pinch of salt: show that, when $\mathfrak{C}$ has nodes, the natural sheaf homomorphism $T_{\mathfrak{X}} \to N_f$ is not surjective.

(21) Find concrete models for the moduli stacks of stable morphisms of degree ≤ 2 from orbi-curves to $\mathbb{P}(1,1,2)$. Determine which components have the correct dimension, which are smooth as Deligne-Mumford stacks, and the nature of all the singular points.

[Degree two is very tough, but try to do at least the case of morphisms of degree 3/2.]

(22) Give a sensible definition of a "stacky" *topological Euler number* of a smooth stack curve. State some properties of the topological Euler number. Let C be a smooth proper curve and G a finite group acting on C: calculate the stacky topological Euler number of the stack $[C/G]$ in terms of vertices, edges and faces of a G-invariant cellular decomposition of C.

(23) (i) Given a Deligne-Mumford stack $\mathfrak{X}$, build a model for the simplicial stack made of moduli stacks $\mathfrak{X}_{0,\bullet,0}$ of genus 0 $\bullet$-pointed stable morphisms of degree 0 in terms of "higher inertia" stacks $\mathfrak{X}_\bullet$. Carefully identify all degeneneracy and face maps.

(ii) Build a model for the moduli stack $\mathfrak{X}_{1,1,0}$ of genus 1 1-pointed stable morphisms of degree 0. Be careful: this is rather tricky. For instance if $\mathfrak{X} = BG$, then $\mathfrak{X}_{1,1,0}$ is a moduli stack of G-twisted covers.

In the next two questions, you are asked to compute the *small quantum orbifold cohomology* of a simple explicit stack $\mathfrak{X}$ naïvely from the definition. This is hard work but it does give a "body" to a very abstract formalism.

If $\mathbf{w} = (w_0, \dots, w_n)$ is an integer vector and $\mathfrak{X} = \mathbb{P}^{\mathbf{w}}$ the corresponding weighted projective space, than the components of the inertia stack are in 1-to-1 correspondence with the set

$$F = \left\{ \frac{k_i}{w_i} \mid i = 0, \dots, n; \quad 0 \le k_i < w_i \right\}$$

We denote by $\mathfrak{X}_{0,n,d}(f_1, \dots, f_n)$ the connected component of $\mathfrak{X}_{0,n,d}$ of stable morphisms which "evaluate" in the components of inertia corresponding to $f_1, \dots, f_n \in F$.

I often confuse degree in cohomology with degree in the Chow ring—please sort out the factors of 2 on your own.

(24) $\mathfrak{X} = \mathbb{P}(1,1,3)$ (i) Show that $H^\bullet_{\text{orb}}\mathfrak{X}$ is generated as a vector space by classes $\mathbf{1}, \eta_{\frac{1}{3}}, A = \mathcal{O}(1), \eta_{\frac{2}{3}}, A^2$ in cohomology degrees $0, 2/3, 1, 4/3, 2$.

(ii) Show directly from the definition that

$$\eta_{\frac{1}{3}} \cup \eta_{\frac{1}{3}} = \eta_{\frac{2}{3}}, \quad \text{and} \quad \eta_{\frac{1}{3}} \cup \eta_{\frac{2}{3}} = A^2 = \frac{1}{3}\text{pt}.$$

[Hint. For the first one look for constant representable morphisms in $\mathfrak{X}_{0,3,0}(\frac{1}{3}^3)$. Note that the last point evaluates with inversion in $\mathbb{P}(3)_{\frac{2}{3}}$. The moduli space has virtual dimension $0 + 2 - 2/3 - 2/3 - 2/3$=0; etc.

For the second look for constant representable morphisms in $\mathfrak{X}_{0,3,0}(1/3, 2/3, 0)$; the expected dimension of the moduli space is $0 + 2 - 2/3 - 4/3 = 0$; the relevant component of the moduli space is isomorphic to $\mathbb{P}(3)$; by definition

$$\eta_{\frac{1}{3}} \cup \eta_{\frac{2}{3}} = e_{3\,*}\mathbf{1} = \frac{1}{3}\text{pt}$$

—whatever it is, it has degree $\int_{\mathbb{P}(3)} \mathbf{1} = 1/3$.]

Finally, it is clear that $\eta_{\frac{2}{3}} \cup \eta_{\frac{2}{3}} = 0$; indeed, $\mathfrak{X}_{0,3,0}(\frac{2}{3}^3)$ has virtual dimension $0 + 2 - 4/3 - 4/3 - 4/3 < 0$.

(iii) First note that $\operatorname{codim} q = \frac{1+1+3}{1\times 1\times 3} = 5/3$ (why?). In the basis above, write down the matrix of quantum multiplication by A:

$$M = \begin{pmatrix} 0 & aq^{\frac{1}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & bq^{\frac{1}{3}} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & cq^{\frac{1}{3}} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and show that $a = b = c = 1/3$ by interpreting the unknown entries a, b, c in terms of stacky Gromov-Witten invariants.

[Hints: First, from $A * \eta_{\frac{1}{3}} = \mathbf{1}aq$ and integrating against A^2:

$$1/3\, aq = \deg A^2\, aq = \langle \mathbf{1}, A^2 \rangle\, aq = \langle A * \eta_{\frac{1}{3}}, A^2 \rangle = \langle A, \eta_{\frac{1}{3}}, A^2 \rangle_{1/3}\, q$$

or

$$a = 3 \int_{\mathfrak{X}_{0,3,1/3}} e_1^*(A) \cup e_2^*(\eta_{\frac{1}{3}}) \cup e_3^*(A^2) \cap e(E).$$

Here $\mathfrak{X}_{0,3,1/3}$ parameterises representable morphisms with image a curve of degree 1/3 on $\mathfrak{X}$; knowing what these curves are, we must be looking at $\mathfrak{X}_{0,3,1/3}(0,1/3,0)$; the virtual dimension is

$$\dim \mathfrak{X}_{0,3,1/3}(0,1/3,0) = 5/3 + 2 - 2/3 = 3.$$

The virtual dimension is the actual dimension and the problem is unobstructed; you can integrate:

$$a = 3 \int_{\mathfrak{X}_{0,3,1/3}} e_1^*(A) \cup e_2^*(\eta_{\frac{1}{3}}) \cup e_3^*(A^2) =$$

$$= \int_{\mathfrak{X}_{0,2,1/3}} e_1^*(A) \cup e_2^*(\eta_{\frac{1}{3}}) = \int_{\mathbb{P}(1,3)} A = \frac{1}{3}$$

Sanity check: $a = 3\langle A, \eta_{\frac{1}{3}}, A^2\rangle_{1/3} = \langle A, \eta_{\frac{1}{3}}, 3A^2\rangle_{1/3} =$ (by the divisor axiom) $= 1/3\langle \eta_{\frac{1}{3}}, \mathrm{pt}\rangle_{1/3} = 1/3$: there is just one orbi-line of degree 1/3 that passes through the singular point and one additional general point. The corresponding stable morphism has no automorphisms, hence this line contributes with "multiplicity 1" to $\langle \eta_{\frac{1}{3}}, \mathrm{pt}\rangle_{1/3}$.

Second, from $A * \eta_{\frac{2}{3}} = bq\, \eta_{\frac{1}{3}}$, we derive

$$\frac{1}{3} b = b\,\langle \eta_{\frac{1}{3}}, \eta_{\frac{2}{3}}\rangle = \langle A, \eta_{\frac{2}{3}}, \eta_{\frac{2}{3}}\rangle_{1/3}$$

The relevant moduli space is $\mathfrak{X}_{0,3,1/3}(1,2/3,2/3)$; it has expected dimension

$$5/3 + 2 - 4/3 - 4/3 = 1.$$

The only way to achieve this is by gluing a morphism in $\mathfrak{X}_{0,3,0}(\frac{2}{3}^3)$ with one in $\mathfrak{X}_{0,2,1/3}(1/3,0)$; the two orbi-curves glue as a nontrivial twisted curve and the map is constant on the first component. This space is two dimensional; we have to deal with a one-dimensional

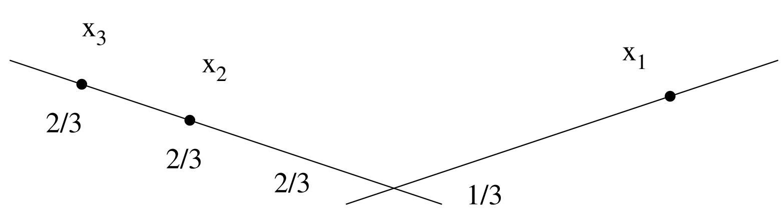

Figure 1: $\mathfrak{X}_{0,3,\frac{1}{3}}(0, \frac{2}{3}, \frac{2}{3})$

obstruction bundle. Note that the first component is in $\mathfrak{X}_{0,3,0}(\frac{2}{3}^3)$ which has negative virtual dimension $0+2-3\times(4/3)=-2$ but it is still there. Fortunately, we can calculate b using the associativity relations:

$$A^2 * \eta_{\frac{2}{3}} = A*(A*\eta_{\frac{2}{3}}) = bqA*\eta_{\frac{1}{3}} = abq^2\mathbf{1}, \quad \text{hence}$$

$$abq^2 = \langle A^2 * \eta_{\frac{2}{3}}, \mathrm{pt}\rangle = \langle A^2, \eta_{\frac{2}{3}}, \mathrm{pt}\rangle_{\frac{2}{3}}\, q^2.$$

We calculate an integral over $\mathfrak{X}_{0,3,2/3}(0,2/3,0)$; the generic point of this moduli space is a stable morphism from a reducible curve with three components: The key thing to keep

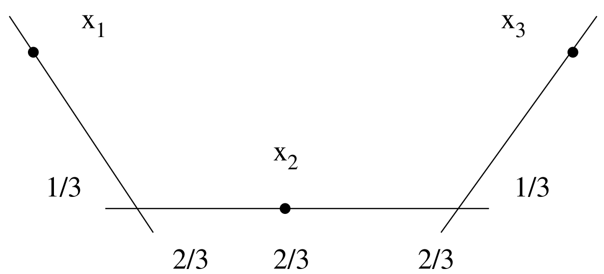

Figure 2: $\mathfrak{X}_{0,3,\frac{2}{3}}\left(0,\frac{2}{3},0\right)$

in mind is that the corresponding morphism always has a μ_3 of automorphisms over $\mathbb{P}(1,1,3)$, coming from the central component $\mathfrak{C}$ on which the morphism is constant. The central component maps to $B\mu_3$ and it carries an induced μ_3-bundle; this bundle has a μ_3 of automorphisms which survive as nontrivial automorphisms of $\mathfrak{C}$ over $B\mu_3$. Having said this, we can now calculate b:

$$ab = \frac{1}{3}b = \frac{1}{3}\int_{\mathfrak{X}_{0,3,2/3}(0,2/3,0)} e_1^*(\mathrm{pt}) \cup e_2^*(\eta_{\frac{2}{3}}) \cup e_3^*(\mathrm{pt}) = \frac{1}{3}\int_{\mathbb{P}(3)} \mathbf{1} = \frac{1}{9}$$

that is, $b=1/3$.

Third, show that $c=a$. Indeed, from $A*A^2 = cq\,\eta_{\frac{2}{3}}$, we get

$$1/3\, cq = cq\deg\langle \eta_{\frac{2}{3}} \cup \eta_{\frac{1}{3}}\rangle = cq\langle \eta_{\frac{2}{3}}, \eta_{\frac{1}{3}}\rangle = \langle A, A^2, \eta_{\frac{1}{3}}\rangle_{1/3}\, q$$

and $c = 3\langle A, A^2, \eta_{\frac{1}{3}}\rangle_{1/3} = \langle A, \mathrm{pt}, \eta_{\frac{1}{3}}\rangle_{1/3} = 1/3\langle \mathrm{pt}, \eta_{\frac{1}{3}}\rangle_{1/3} = 1$ as before.]

(iv) Let $D = q\frac{d}{dq}$ and consider the *quantum differential equation*

$$D\Psi = \Psi M \quad \text{for} \quad \Psi\colon \mathbb{C}^\times \to \mathrm{End}\, H^\bullet_{\mathrm{orb}}(\mathfrak{X},\mathbb{C}).$$

In the given basis, write $\Psi = (\psi_0,\dots,\psi_n)$ where ψ_i are column vectors; find the ordinary differential equation satisfied by ψ_0.

[Hint.

$$\begin{aligned}3^3(D-2/3)(D-1/3)D^3\psi_0 &= 3^3(D-2/3)(D-1/3)D^2\psi_2 = \\ 3^3(D-2/3)(D-1/3)D\psi_4 &= 3^2(D-2/3)(D-1/3)q^{1/3}\psi_3 = \\ 3^2q^{1/3}(D-1/3)D\psi_3 &= 3q^{1/3}(D-1/3)q^{1/3}\psi_1 = \\ &3q^{2/3}D\psi_1 = q\psi_0]\end{aligned}$$

(25) $\mathfrak{X} = X_3^2 \subset \mathbb{P}(1^3, 2)$ (i) Note that $\mathfrak{X}$ can be written as

$$\big(yx_0 + a_1(x_2, x_3) = 0\big) \subset \mathbb{P}(1^3, 2)$$

Build a mental picture of $\mathfrak{X}$ by studying the obvious birational map $\mathfrak{X} \dashrightarrow \mathbb{P}^2$: $\mathfrak{X}$ is obtained by blowing up three collinear points and contracting the proper transform of the line $(x_0 = 0)$. Let $A = \mathcal{O}_{\mathfrak{X}}(1)$. In particular, on $\mathfrak{X}$, there are:

- Three 'lines' of A-degree $1/2$;
- Three fibrations by 'conics' of A-degree 1;
- One map to $\mathbb{P}^2$.

(ii) Convince yourself that the orbifold cohomology of $\mathfrak{X}$ has basis $\mathbf{1}, A, \eta, A^2$ in degrees $0, 1, 1, 2$; $\deg A^2 = 3/2$ and $\deg \eta^2 = 1/2$.

(iii) Use the information in (i) to show that quantum multiplication by A is given in this basis by the following matrix where $\operatorname{codim} q = 2$:

$$M = \begin{pmatrix} 0 & q\langle A, A, \mathrm{pt}\rangle_1 & 0 & 0 \\ 1 & 0 & \frac{2}{3}q^{\frac{1}{2}}\langle A, \eta, A\rangle_{\frac{1}{2}} & q\langle A, \mathrm{pt}, A\rangle_1 \\ 0 & 2q\langle A, A, \eta\rangle_{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 3q & 0 & 0 \\ 1 & 0 & \frac{1}{2}q^{\frac{1}{2}} & 3q \\ 0 & \frac{3}{2}q^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(iv) Verify directly that the cyclic vector ψ_0—cf. Q 8(4)—satisfies the 'expected' hypergeometric operator

$$2D^3(2D-1) - 3q(3D+1)(3D+2).$$

[Hint: First rewrite $D\psi = \psi M$ in the new basis

$$\begin{aligned} \phi_0 &= \psi_0 \\ \phi_1 &= D\psi_0 = \psi_1 \\ \phi_2 &= \psi_3 + \frac{3}{2}q^{\frac{1}{2}}\psi_2 \\ \phi_3 &= \psi_2 \end{aligned}$$

A small calculation shows that the equation in the new basis is:

$$D\Phi = \Phi \begin{pmatrix} 0 & 3q & 0 & 0 \\ 1 & 0 & \frac{15}{4}q & \frac{1}{2}q^{\frac{1}{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4}q^{\frac{1}{2}} & 0 \end{pmatrix}$$

In this form it is easy to calculate the equation satisfied by $\phi_0 = \psi_0$.]

(v) Show that the quantum products calculated in (iii), together with associativity, determine the whole small quantum cohomology ring. In particular, show that this determines the curious Gromov-Witten number:

$$\langle \eta, \eta, \eta \rangle_{\frac{1}{2}} = -\frac{3}{4}.$$

(vi) The direct calculation of the number in (v) leads to a beautiful case study in excess intersection theory: the expected dimension of the moduli space is $1+2-3\times 1 = 0$;

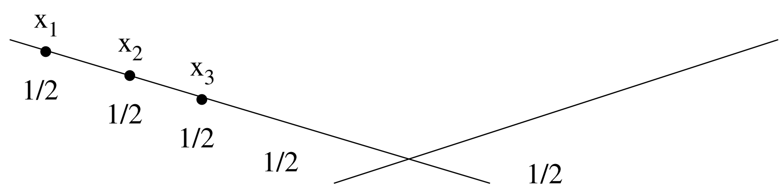

Figure 3: $\mathfrak{X}_{0,3,1/2}\left(\frac{1}{2}^3\right)$

however, the picture shows an actual moduli space of dimension 1 (the four points on the component on which the morphism to X is constant). If you feel brave enough, calculate $\langle \eta, \eta, \eta \rangle_{\frac{1}{2}} = -\frac{3}{4}$ by a study of the virtual class.

6 Conjecture B

(26) Verify Conjecture B for the following surfaces taken from the 26 surfaces in Table 2:

(i) No. 26, extended to all of $H^{<2}_{\mathrm{orb}}(X, \mathbb{C}) \oplus \mathbb{C}(-K_X)$.

(ii) No. 22, nonextended, with mirror

$$f(x, y) = \frac{(1+y)^3}{xy} + \frac{1}{y} + xy$$

(iii) No. 15, nonextended, with mirror

$$f(x, y) = \frac{(1+x)^2(1+y)^2}{xy} + \frac{(1+y)^2}{x} + 2y - 4$$

(iv) No. 4, nonextended, with mirror

$$f(x, y) = \left(1 + \frac{1}{x} + \frac{1}{y}\right)^4 \left(\frac{y}{x^2} + \frac{y^2}{x}\right) - 24$$

(27) In this question we verify Conjectures A and B in a simple 3-fold situation:

(i) Find all the rigid maximally mutable Laurent polynomials on the 3D Fano polytope with vertex matrix:

$$\begin{pmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

You should find precisely two such polynomials:

$$f(x,y,z) = xz^{-1} + yz^{-1} + 2x^{-1}y^{-1}z^{-1} + x^{-2}y^{-3}z^{-1} + 3x^{-1}y^{-2}z^{-1} + 3y^{-1}z^{-1} + z + 2z^{-1}$$
$$g(x,y,z) = xz^{-1} + yz^{-1} + 2x^{-1}y^{-1}z^{-1} + x^{-2}y^{-3}z^{-1} + 3x^{-1}y^{-2}z^{-1} + 3y^{-1}z^{-1} + z + 3z^{-1}$$

(ii) Compute the first few terms of the quantum period and the Picard–Fuchs differential operators for these two polynomials. (Don't be too proud: go and get some help: ask someone with a computer):

$$L_f = -D^3 + 4t^2(D+1)(7D^2+14D+8) + t^4 128(D+1)(D+2)(D+3)$$
$$L_g = -D^3 + 8t^2(D+1)(5D^2+10D+6) - t^4 144(D+1)(D+2)(D+3)$$

(iii) Show that $\mathrm{Vol}(P^\star) = 48$. Go on a table of Fano 3-folds and find that there are two families with $K^2 = 48$. Show that f is mirror to $X_{(1,1)} \subset \mathbb{P}^2 \times \mathbb{P}^2$; and g is mirror to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(28) Find all the rigid maximally mutable Laurent polynomials on the 4D Fano polytope with vertex matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

You should just find

$$f(x,y,z,w) = x + y + z + w + \frac{2}{xyz} + \frac{1}{x^2y^2z^2w}$$

(ii) Compute the first few terms of the quantum period and the Picard–Fuchs differential operator:

$$-D^4 + 124t^4(D+1)(D+2)^2(D+3)$$

(iii) Show that f is mirror to the 4-dimensional quadric.

(29) Find all the rigid maximally mutable Laurent polynomials on the 4D Fano polytope with vertex matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}$$

You should just find

$$f(x,y,z,w) = x + y + z + w + \frac{z}{xyw} + \frac{1}{xy} + \frac{1}{xyw} + \frac{1}{xyz}$$

(ii) Compute the first few terms of the quantum period and the Picard–Fuchs differential operator.

(iii) Show that f is mirror to the 4-dimensional toric hypersurface X of type $(2,1)$ in the toric variety F with weight data:

$$\begin{matrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{matrix}$$

(what IS this variety?)

7 Appendix

Table 2: Complete intersection descriptions of the representatives of Table 1 for the 26 mutation-equivalence classes with singularity content $(n, \{m \times \frac{1}{3}(1,1)\})$.

#	Weights and line bundles
1	$\begin{array}{cccc\|c} 1 & 2 & 3 & 5 & 10 \end{array}$
2	$\begin{array}{ccccc\|cc} 1 & 3 & 3 & 2 & 2 & 6 & 4 \end{array}$
3	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 & 2 & 4 \end{array}$
4	$\begin{array}{ccccc\|c} 1 & 0 & 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 & 1 & 4 \end{array}$
5	$\begin{array}{cccccc\|cc} 1 & 0 & 2 & 2 & 1 & 1 & 4 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 & 4 \end{array}$

6	$\begin{array}{ccccc\|c} 1 & 0 & 1 & 1 & 2 & 4 \\ 1 & 1 & 0 & 3 & 3 & 6 \end{array}$
7	$\begin{array}{ccccccccc\|cc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & -2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 2 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & 0 \end{array}$
8	$\begin{array}{cccccc\|c} 1 & 0 & 1 & 0 & 3 & 2 & 5 \\ 0 & 1 & 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \end{array}$
9	none
10	$\begin{array}{ccccccc\|c} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 4 & 0 & 0 & 4 \end{array}$
11	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 3 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 & 3 & 0 & 3 \\ 2 & 0 & 1 & 3 & 3 & 0 & 6 \end{array}$
12	$\begin{array}{cccc\|c} 1 & 3 & 3 & 1 & 6 \end{array}$
13	$\begin{array}{ccccc\|c} 1 & 0 & 1 & 1 & 1 & 3 \\ 1 & 1 & 3 & 0 & 0 & 3 \end{array}$
14	$\begin{array}{ccccc\|c} 1 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 0 & 0 & 3 \end{array}$
15	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 1 & 3 & 4 \end{array}$
16	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 4 & 0 & 0 & 4 \end{array}$

<table>
<tr><td>17</td><td>$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{array}$</td></tr>
<tr><td>18</td><td>$\left.\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right.$</td></tr>
<tr><td>19</td><td>$\left.\begin{array}{cccccc|c} 1 & 1 & 0 & 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 5 & 0 & 0 & 5 \end{array}\right.$</td></tr>
<tr><td>20</td><td>$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 3 \end{array}$</td></tr>
<tr><td>21</td><td>$\left.\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 4 \end{array}\right.$</td></tr>
<tr><td>22</td><td>$\left.\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right.$</td></tr>
<tr><td>23</td><td>$\left.\begin{array}{ccccc|c} 1 & 0 & 3 & -1 & 1 & 2 \\ 2 & 1 & 3 & 1 & 0 & 4 \end{array}\right.$</td></tr>
<tr><td>24</td><td>$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 3 \end{array}$</td></tr>
<tr><td>25</td><td>$\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}$</td></tr>
<tr><td>26</td><td>$\begin{array}{ccc} 1 & 1 & 3 \end{array}$</td></tr>
</table>

Table 1: Representatives for the 26 mutation-equivalence classes of Fano polygons $P \subset N_{\mathbb{Q}}$ with singularity content $(n, \{m \times \frac{1}{3}(1,1)\})$. The degrees $-K_X^2 = 12 - n - \frac{5m}{3}$ of the corresponding toric varieties are also given. See also Figure 4.

#	$\mathcal{V}(P)$	n	m	$-K_X^2$
1	$(7,5),(-3,5),(-3,-5)$	10	1	$\frac{1}{3}$
2	$(3,2),(-3,2),(-3,-2),(3,-2)$	8	2	$\frac{2}{3}$
3	$(3,1),(3,2),(-1,2),(-2,1),(-2,-3),(-1,-3)$	6	3	1
4	$(3,2),(-1,2),(-2,1),(-2,-3)$	9	1	$\frac{4}{3}$
5	$(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2),(1,-2),(2,-1)$	4	4	$\frac{4}{3}$
6	$(3,2),(-1,2),(-2,1),(-2,-1),(-1,-2)$	7	2	$\frac{5}{3}$
7	$(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2),(1,-1)$	2	5	$\frac{5}{3}$
8	$(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2)$	5	3	2
9	$(1,1),(-1,2),(-2,1),(-1,-1),(1,-2),(2,-1)$	0	6	2
10	$(1,1),(-1,2),(-1,-2),(1,-2)$	8	1	$\frac{7}{3}$
11	$(1,1),(-1,2),(-2,1),(-1,-1),(2,-1)$	3	4	$\frac{7}{3}$
12	$(3,1),(-3,1),(0,-1)$	6	2	$\frac{8}{3}$
13	$(1,1),(-1,2),(-1,-1),(2,-1)$	6	2	$\frac{8}{3}$
14	$(1,1),(-1,2),(-2,1),(-1,-1),(1,-1)$	4	3	3
15	$(1,1),(-1,2),(-1,-1),(1,-1)$	7	1	$\frac{10}{3}$
16	$(1,1),(-1,2),(-1,0),(0,-1),(2,-1)$	5	2	$\frac{11}{3}$
17	$(1,0),(1,1),(-1,2),(-2,1),(-1,-1),(0,-1)$	3	3	4
18	$(1,0),(0,1),(-1,1),(-1,-3)$	6	1	$\frac{13}{3}$
19	$(1,1),(-1,2),(-1,1),(0,-1),(2,-1)$	4	2	$\frac{14}{3}$
20	$(1,1),(-1,2),(-2,1),(-1,-1),(0,-1)$	2	3	5
21	$(1,1),(-1,2),(-1,-2)$	5	1	$\frac{16}{3}$
22	$(1,1),(-1,2),(-1,-1),(0,-1)$	5	1	$\frac{16}{3}$
23	$(1,1),(-1,2),(0,-1),(2,-1)$	3	2	$\frac{17}{3}$
24	$(0,1),(-1,2),(-2,1),(-1,0),(1,-1)$	4	1	$\frac{19}{3}$
25	$(0,1),(-1,2),(-2,1),(1,-1)$	3	1	$\frac{22}{3}$
26	$(-1,2),(-2,1),(1,-1)$	2	1	$\frac{25}{3}$

Figure 4: Representatives for the 26 mutation-equivalence classes of Fano polygons $P \subset N_{\mathbb{Q}}$ with singularity content $(n, \{m \times \frac{1}{3}(1,1)\})$. See also Table 1.