Algebraic Topology M3P21 2015 solutions 2

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A small disclaimer

This document is a bit sketchy and it leaves some to be desired in several other respects too. I thought it is more useful to you if I show you this now than if I show you a much better document at a time infinitely far into the future.

(1) The equivalence of (i) and (ii) is an immediate consequence of Problem 7 of sheet 1.

The key point is this: fix $x_0 \in X$ and let $f\colon S^1 \to X$ be a loop based at x_0. Assuming that we have $F\colon D^2 \to X$ a continuous map such that $F_{|S^1} = f$, we want to show that the path $\gamma(t) = f(e^{2\pi i t})$ is homotopic to the constant path e_{x_0} *as loops*, that is rel $\{0,1\}$. All you have to do is to produce an explicit continuous map $\Phi\colon I \times I \to D^2 \subset \mathbb{C}$ such that:

(i) $\Phi(t,0) = e^{2\pi i t}$, and

(ii) for all t, $s \in I$, $\Phi(t,1) = \Phi(0,s) = \Phi(1,s) = 1$

(DO write down explicitly such a map!) Then $\Gamma = F \circ \Phi\colon I \times I \to X$ is the desired path homotopy from γ to e_{x_0}.

(2) (a) Informally it suffices to substitute $x = a$ and $y = a^{-1}b$. Then $x^2y^2 = aba^{-1}b$, and we only need to observe that the substitution can be reversed by setting $a = x$ and $b = xy$.

More formally, let $F = \langle u, v \rangle$ be the free group on two generators. By definition, $G_i = F/N_i$, with N_1 (N_2) the smallest normal subgroup of F containing the element $uvu^{-1}v$ (u^2v^2) respectively. We then define a homomorphism $\phi : F \to F$ by setting $\phi(u) = u$ and $\phi(v) = uv$. (Homomorphisms of free groups can be "prescribed on generators" just like linear maps of vector spaces can be prescribed on basis vectors. This is made rigorous by the universal property of free groups.) This is in fact an isomorphism with inverse given by $\psi(u) = u$, $\psi(v) = u^{-1}v$. It is now easy to see that $\phi(N_1) = N_2$ and $\psi(N_2) = N_1$, so that G_1, G_2 are isomorphic via $gN_1 \mapsto \phi(g)N_2$ and $gN_2 \mapsto \psi(g)N_1$.

(b) $\pi_1(M) = \mathbb{Z} = \langle \alpha \rangle$, where α is the homotopy class of the "soul" S of the Möbius strip, i.e. the image of $[0,1] \times \{\frac{1}{2}\}$ under the usual quotient map $[0,1] \times [0,1] \to M$, where $(0,y) \sim (1, 1-y)$. This is clear by observing that the map $r : M \to S$ given by $r([(x,y)]) = [(x, \frac{1}{2})]$ is a strong deformation retraction. (We don't have to worry about orientations: If α is a generator then so is α^{-1}.)

Key observation: ∂M is connected, i.e. a circle, and $[\partial M] = \alpha^2$ in $\pi_1(M)$. This is almost entirely analogous to the reasoning I used to convince you that $\pi_1(\mathbb{P}^2(\mathbb{R})) = \mathbb{Z}_2$ viewing $\mathbb{P}^2(\mathbb{R})$ as $B^2/\sim$.

Van Kampen now tells us that $\pi_1(K) = \langle \alpha, \beta \,|\, \alpha^2 = \beta^2 \rangle$, which modulo replacing α or β by its inverse is clearly the same as the second presentation from (a).

(3) (a) There are many retractions $r : S^1 \vee S^1 \to S^1$. E.g. we can keep the first factor pointwise fixed and collapse the second factor onto the usual basepoint. More formally, recall that

$$S^1 \vee S^1 = [(S^1 \times \{0\}) \cup (S^1 \times \{1\})]/\{(x_0, 0), (x_0, 1)\}$$

for some (any) given point $x_0 \in S^1$. Define a map

$$\hat{r} : (S^1 \times \{0\}) \cup (S^1 \times \{1\}) \to S^1$$

by $\hat{r}(x, 0) := x$ and $\hat{r}(x, 1) := x_0$. This is obviously continuous, and constant on equivalence classes, and hence factors through a continuous map $\tilde{r} : S^1 \vee$

$S^1 \to S^1$. Now define a map $i : S^1 \to S^1 \vee S^1$ by $i(x) := [(x, 0)]$. Then i is a homeomorphism onto its image, and $r := i \circ \tilde{r}$ does the job.

If S^1 was a deformation retract of $S^1 \vee S^1$, then their fundamental groups would be isomorphic by a proposition from lectures, but we also know that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, which aren't isomorphic as groups if only because they aren't both abelian.

Remark: Notice that the inclusion $S^1 \to S^1 \vee S^1$ induces an injection on π_1 for concrete reasons ($\langle a \rangle \to \langle a, b \rangle$) as well as abstract reasons (existence of a retraction).

(b) Say $i \colon S^1 \hookrightarrow S^1 \vee S^1$ is the inclusion of the original circole into the first circle. For all $n \in \mathbb{Z}$, define a retraction $r_n \colon S^1 \vee S^1 \to S^1$ such that the first circle is mapped back identically to the original circle, and the second circle cover the original circle n times. I leave it to you to show rigorously that if $n \neq m$ then $r_n \not\simeq r_m$.

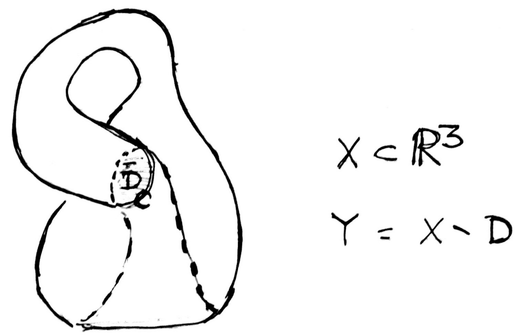
(4) The statement really isn't true for *arbitrary* decompositions $X = U \cup V$ even with $U \cap V$ connected, and there are easy counterexamples where U is open and V is closed: Take $X = S^1$, let $p : [0, 1] \to X$ be the usual quotient map ($0 \sim 1$), $U = p((0, 1))$, $V = p([0, \frac{1}{2}])$. Then $U \cup V = X$; the intersection $U \cap V = p((0, \frac{1}{2}])$ is connected; and all three pieces are simply-connected.

Remark: The amalgamated product in this example is *too small* compared to $\pi_1(X)$, so the part of the proof that fails is the "easy" step that I sketched in class (surjectivity of Φ).

(5) Let $Z \subset \mathbb{R}^n$ be a discrete subset. The statement is more or less obvious if Z is finite. If Z is infinite, it still works: for all $z \in Z$, choose a small punctured ball $B(z, \varepsilon_z)^\star \subset X$: then $\mathbb{R}^n$ is obtained from X by attaching all the $B(z, \varepsilon_z)$.

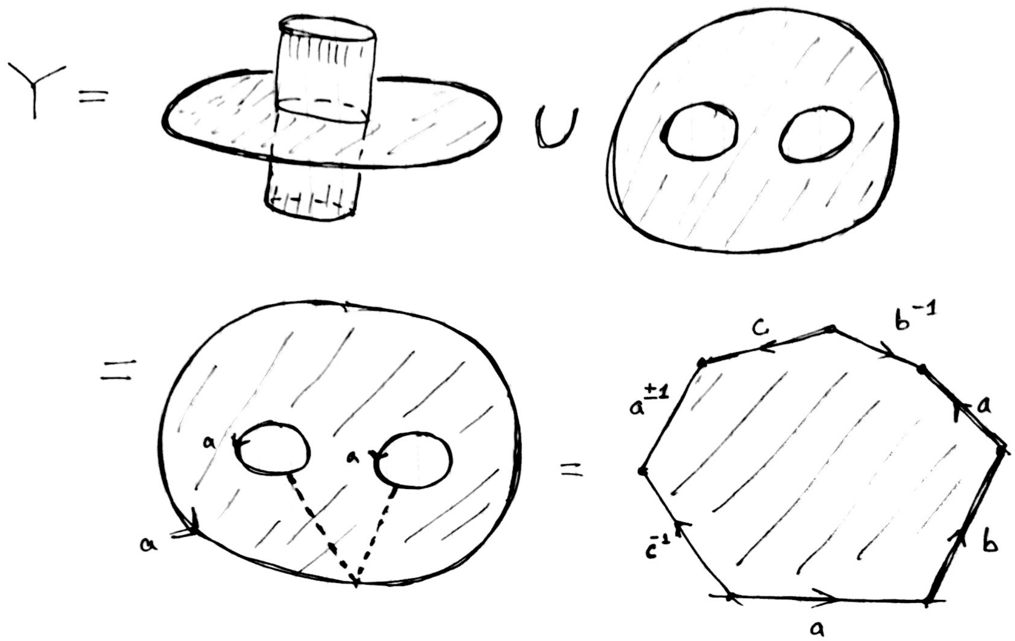
(6) This was meant to be a tough question and I only sketch the key ideas here.

Here is a drawing of $X \subset \mathbb{R}^3$:



By shrinking the disk $D \subset X$ to a point, we see that X is homotopic to $S^2 \vee S^1 \vee S^1$ and hence $\pi_1(X) = \mathbb{Z} \star \mathbb{Z}$ as claimed.

This drawing shows how to construct Y:



It is clear from the drawing that Y is obtained from identifying the 3 boundary circles on a disk with two holes or, equivalently, the sides of a 7-gon as drawn. There are two ways to to do this upto homeomorphism:

(i) taking a^{-1} in the drawing gives Y and

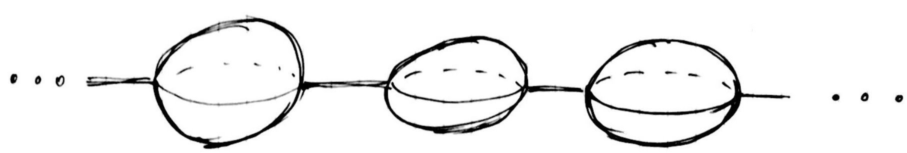
$$\pi_1(Y) = G_1 = \langle a, b, c \mid abab^{-1}ca^{-1}c^{-1} \rangle$$

(ii) taking a in the drawing gives a topological space Z with

$$\pi_1(Z) = G_2 = \langle a, b, c \mid abab^{-1}cac^{-1} \rangle$$

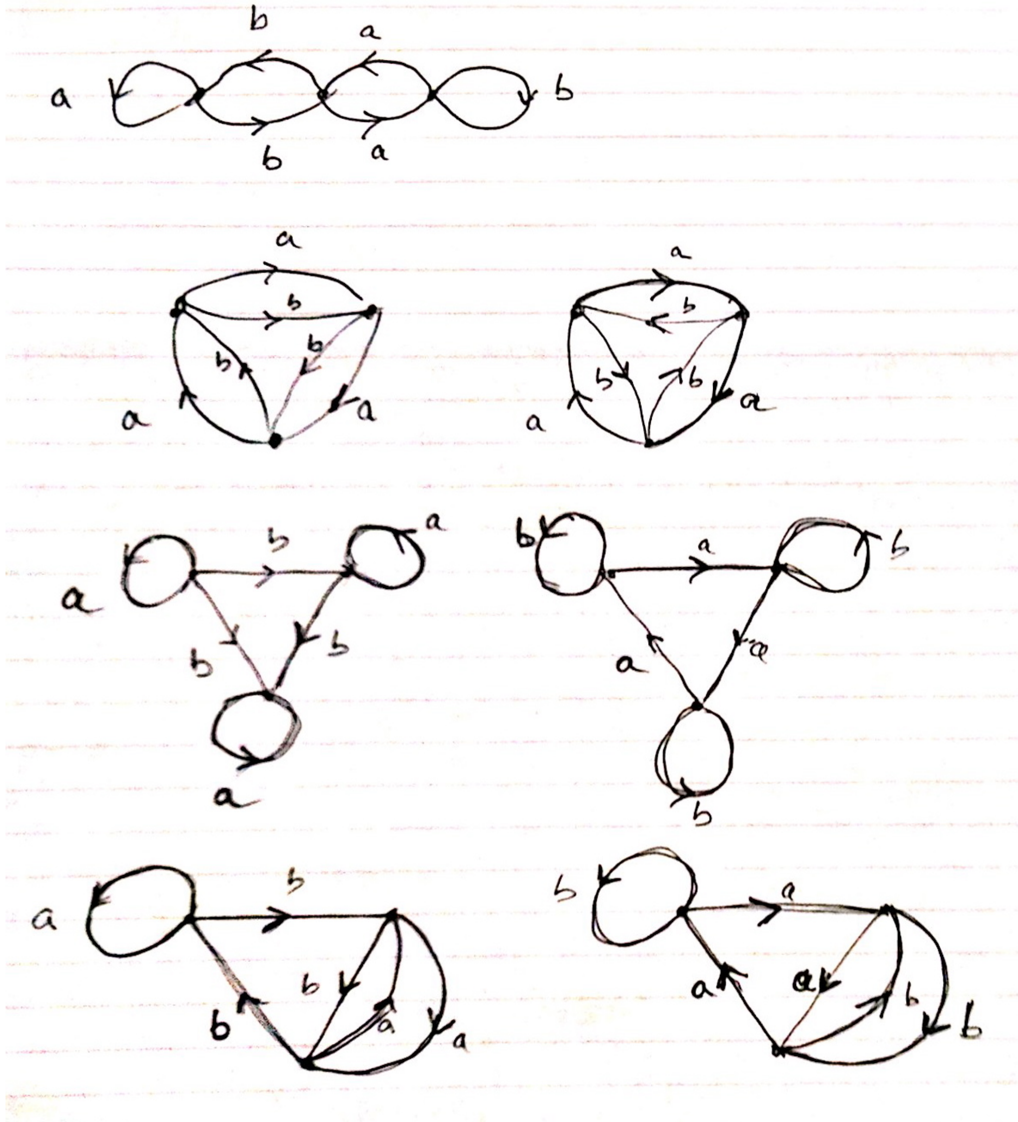
It is clear that Y and Z are not homeomorphic: indeed G_1 and G_2 are not isomorphic; indeed $G_1^{\text{ab}} = \mathbb{Z}^2$ is not isomorphic to $G_2^{\text{ab}} = \mathbb{Z}^2 \oplus (\mathbb{Z}/3\mathbb{Z})$.

(7) The first is not hard:



The second is tricky to describe and draw: I will do it in class if you ask me.

(8) We know that the set of covers of $S^1 \vee S^1$ upto isomorphism is the set of 2-oriented 2-valent graphs upto isomorphism. I leave the 2-sheeted covers to you (we did this in class). The 3-sheeted covers are the graphs with 3 vertices:



(9) I only give the basic idea here. Choose a point $P \in \mathbb{P}^2(\mathbb{R})$. If X is a covering of $\mathbb{P}^2(\mathbb{R})$ then:

Either $X = \mathbb{P}^2(\mathbb{R}) \cup (\cup^n S^2) \cup \mathbb{P}^2(\mathbb{R})$ is the following space: first attach a chain of n spheres (n=0 is allowed) by identifying the North pole of the i-th sphere with the South pole of the $i+1$-st sphere, then attach a copy of $\mathbb{P}^2(\mathbb{R})$ by identifying P with the South pole of the first sphere, and finally attach a second copy of $\mathbb{P}^2(\mathbb{R})$ by identifying P with the North pole of the last sphere.

Or $X = \cup^n S^2$ is the space obtained by attaching $n \geq 2$ spheres in a circle by identifying the North pole of the i-th sphere with the South pole of the $i+1$-st sphere and the North pole of the last sphere with the South pole of the first sphere.

(10) Recall that the commutator subgroup of a group G is the normal subgroup of $[G,G] \leq G$ generated by all commutators $[a,b] = aba^{-1}b^{-1}$ of two elements $a, b \in G$; the quotient $G/[G,G] = G^{\mathrm{ab}}$ is an abelian group, the *abelianization* of G.

The covering in question is the covering $p\colon (X^{\mathrm{ab}}, x_0) \to (X, x)$ corresponding to the commutator subgroup $H \leq \pi_1(X, x)$ under the fundamental theorem of covering space theory.

Suppose that $q\colon (Y, y) \to (X, x)$ is an abelian cover. Under the correspondence of the fundamental theorem, this cover corresponds to a normal subgroup $H = q_\star \pi_1(Y, y) \leq G$ with abelian quotient group G/H. Because the quotient is abelian, necessarily $[G,G] \leq H$. By the lifting theorem (b) $p\colon (X^{\mathrm{ab}}, x_0) \to (Y, y)$ lifts as a (base-point preserving) continuous map $\tilde{p}\colon X^{\mathrm{ab}} \to Y$, which is itself a covering, and this shows that X^{ab} is a covering space of every other abelian cover of X.

The universal abelian cover of $S^1 \vee S^1$ is homeomorphic to the standard square mesh in $\mathbb{R}^2$.