Algebraic Topology M3P21 2015
Homework 4

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N.B.

This example sheet will NOT be assessed. All questions are copied from Hatcher: please do them for your own good.

(1) Show that any two reflections in S^n across different n-dimensional hyperplanes are homotopic, in fact homotopic through reflections. [The linear algebra formula for a reflection in terms of innner products may be helpful.]

(2) For an invertible linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ show tht the induced map on $H_n\big(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}\big) = \mathbb{Z}$ is $\mathbf{1}$ or $-\mathbf{1}$ according to whether the determinant of f is positive or negative. [Use Gaussian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with ± 1 on the diagonal.]

(3) Compute the homology groups of the following 2-dimensional CW complexes:

(a) $S^1 \times (S^1 \vee S^1)$;

(b) the space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles;

(c) the quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $\frac{2\pi}{m}$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $\frac{2\pi}{n}$ rotation.

(4) A map $f\colon S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an even map. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show that there exist even maps of any given even degree. [Hints: if n is even, it factors as a composition $S^n \to \mathbb{P}^n(\mathbb{R}) \to SS^n$. What is $H_n\,\mathbb{P}^n(\mathbb{R})$?. Show that if n is odd the induced map $H_n\,S^n \to H_n\,\mathbb{P}^n(\mathbb{R})$ sends a generator to twice a generator. It may be helpful to show that when n is odd the quotient map $\mathbb{P}^n(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})/\mathbb{P}^{n-1}(\mathbb{R})$ induces an isomorphism on H_n.]

(5) For $m < n$, compute $H_i\,\mathbb{P}^n(\mathbb{R})/\mathbb{P}^m(\mathbb{R})$ by cellular homology, using the standard CW structure on $\mathbb{P}^n(\mathbb{R})$ with $\mathbb{P}^m(\mathbb{R})$ as its m-skeleton.

(6) For finite CW complexes X, Y, prove that $\chi(X \times Y) = \chi(X)\chi(Y)$.

(7) If a finite CW complex is union of subcomplexes A and B, show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

(8) Show that if the closed orientable surface of genus g M_g (a pretzel with g "holes") is a covering space of M_h, then if n is the number of sheets:

$$g = n(h-1) + 1$$

Conversely, if for some positive integer n $g = n(h-1)+1$, construct a n-sheeted covering map $p\colon M_g \to M_h$.

(9)

(a) Use the Mayer–Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$.

(b) Do the same for the space obtained by attaching a Möbius band to $\mathbb{P}^2(\mathbb{R})$ via a homeomorphism of its boundary circle to $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$.

(10) Go and read Example 2.48 in Hatcher. In there he establishes a long exact sequence computing the homology of the mapping torus M_f of a map $f\colon X \to X$:

$$\cdots H_n\, X \xrightarrow{\mathbf{1}-f_\star} H_n\, X \to H_n\, M_f \to H_{n-1}\, X \to \cdots$$

Use this to compute the homology of the mapping tori of the following maps:

(a) A reflection $S^2 \to S^2$;

(b) A map $S^2 \to S^2$ of degree 2;

(c) The map $S^1 \times S^1 \to S^1 \times S^1$ that is the identity on one factor and a reflection on the other;

(d) The map $S^1 \times S^1 \to S^1 \times S^1$ that is a reflection on each factor;

(e) The map $S^1 \times S^1 \to S^1 \times S^1$ that interchanges the two factors and then reflects one of the factors.