Algebraic Topology M3P21 2015
Homework 2

AC
Imperial College London
a.corti@imperial.ac.uk

2nd February 2015

N.B.

Turn in 5 questions by Monday, 16 February, at 12:00 either in class or in my pigeon-hole in the mail-room on the 6th floor.

(1) Show that for a space X the following are equivalent:

(i) Every map $S^1 \to X$ is homotopic to a constant map, with image a point in X;

(ii) Every map $S^1 \to X$ extends to a map $D^2 \to X$;

(iii) $\pi_1(X, x_0) = (0)$ for all $x_0 \in X$

Deduce that a space X is simply-connected if and only if all maps $S^1 \to X$ are homotopic. [N.B. In this question "homotopic" means "homotopic without regard to basepoints."]

(2) This question is about the fundamental group of the Klein bottle K.

(a) Hatcher (page 51) shows two different ways of computing $\pi_1(K)$. The two presentations he obtains are $G_1 = \langle a, b | aba^{-1}b \rangle$ and $G_2 = \langle x, y | x^2y^2 \rangle$. Write a purely algebraic proof that G_1 and G_2 are isomorphic. (Hint: By the definition of "group presentation", $G_i \cong F/N_i$, where $F = \langle u, v \rangle$ is free and N_1, N_2 are certain normal subgroups. Hence it suffices to find group homomorphisms $\phi, \psi : F \to F$ such that $\phi \circ \psi = \psi \circ \phi = \mathrm{id}$ and $\phi(N_1) = N_2$.)

(b) In the first homework we saw that K can also be written as $M \cup_f M$, where M is a Möbius strip and $f : \partial M \to \partial M$ is a homeomorphism. Apply van Kampen to this decomposition to compute $\pi_1(K)$ for the third time. (Hint: $[\partial M]$ is an element of $\pi_1(M)$. Which one?)

(3) Show *very carefully* that S^1 is a retract of $S^1 \vee S^1$, but not a deformation retract.

Construct infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$.

(4) Van Kampen's theorem talks about decompositions $X = U \cup V$, where U, V are open and path-connected, and $U \cap V \neq \emptyset$ is path-connected as well. Show that the assumption that both U and V are open is necessary for the theorem to hold.

(5) Suppose that a space Y is obtained from a path-connected subspace X by attaching n-cells for a fixed $n \geq 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism on π_1. Apply this to show that the complement of a discrete subspace of $\mathbb{R}^n$ is simply-connected if $n \geq 3$. [N.B. a subspace $Z \subset X$ of a topological space X is *discrete* if the topology on Z induced by the topology of X is the discrete topology: in other words, $\forall\, z \in Z$ there is $U \subset X$ open, $z \in U$, $U \cap Z = \{z\}$.]

(6) Recall the usual picture of the Klein bottle K as a subspace $X \subset \mathbb{R}^3$ with a circle of self-intersection (so in fact there is a continuous map $K \to X$ identifiying two circles). If one wanted a model that could actually function as a bottle, one would delete the small open disk bounded by the circle of self-intersection, producing a subspace $Y \subset K$. Show that $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$ and that

$$\pi_1(Y) = \left\langle a, b, c \mid aba^{-1}b^{-1}cb^{\varepsilon}c^{-1} \right\rangle$$

for $\varepsilon = \pm 1$. (Don't worry about nailing down ε.)

The space Y can be obtained from a disk with two holes by identifying the three boundary circles. Show that the other way yields a space Z with $\pi_1(Z)$ not isomorphic to $\pi_1(Z)$. [Hint. In fact, the abelianizations of these groups are not isomorphic.]

(7) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

(8) Draw all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

(9) Find all the connected covering spaces of $\mathbb{P}^2(\mathbb{R}) \vee \mathbb{P}^2(\mathbb{R})$.

(10) For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $\widetilde{X} \to X$ *abelian* if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X.

Draw a picture of this covering space for $X = S^1 \vee S^1$. (No proof is required, just a picture.) [*Hint: paint* $S^1 \vee S^1$ *on the torus.*]