Algebraic Topology M3P21 2015
Homework 1

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N.B.

Turn in 5 questions by Monday, 2 February, at 12:00 either in class or in my pigeon-hole on the 6th floor.

(1) Suppose that $f\colon X \to Y$ be a quotient map of topological spaces.

(a) Show that if Y is Hausdorff, then the fibers $f^{-1}(y)$ ($y \in Y$) are closed.

(b) Is Y necessarily Hausdorff if all the fibers are closed?

(2) Let X, Y be topological spaces, $A \subset X$ a subspace, and $f\colon A \to Y$ a quotient map. Show that the two definitions of $X \cup_f Y$ are equivalent: in other words, $(X \sqcup Y)/R$ is homeomorphic to $X/\sim$, where R is the equivalence relation on $X \sqcup Y$ generated by $x \, R \, f(x)$ for all $x \in A$, and $\sim$ is the equivalence relation on X generated by $x_1 \sim x_2$ for all $x_1, x_2 \in A$ for which $f(x_1) = f(x_2)$.

(3) Show that the quotient of $\mathbb{R} \times \{0,1\}$ by the equivalence relation generated by $(x,0) \sim (\frac{1}{x}, 1)$ for all $x \neq 0$ is homeomorphic to $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

(4) We will take it for granted that the Klein bottle K is homeomorphic to the quotient space $([0,1] \times [0,1])/\sim$, where $\sim$ is generated by $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$.

Using this fact, draw pictures to convince me that K can be written as two Möbius strips M_1, M_2 attached to each other along their boundaries (formally: as the quotient of the disjoint union $M_1 \sqcup M_2$ by the equivalence relation generated by $x \sim f(x)$ for $x \in \partial M_1$, where $f\colon \partial M_1 \to \partial M_2$ is a certain continuous map—in fact, a homeomorphism—that you don't need to specify).

(5) (This is very similar to (4)). Show that one of the three equivalent constructions (given in class) of $\mathbb{P}^2(\mathbb{R})$ is homeomorphic to a Möbius strip M attached to a disk D^2 attached along their boundaries. Here, try to be as specific as possible in defining the attachment, and the homeomorphism as well as its inverse.

(6) We defined in class $\mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} \setminus \{\mathbf{0}\})/\mathbb{C}^\times$. Briefly argue that $\mathbb{P}^n(\mathbb{C}) = S^{2n+1}/\sim$, where $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid |z_1|^2 + \ldots + |z_{n+1}|^2 = 1\}$ and $\sim$ refers to the usual action of the group S^1 of unit complex numbers. Write $p_n\colon S^{2n+1} \to \mathbb{P}^n(\mathbb{C})$ for the quotient map. Also denote by $H^n_{\mathbb{C}} = \{z \in S^{2n+1} \mid z_{n+1} \in [0,\infty)\}$ the complex hemisphere.

(a) Show that the restriction of p_n to $H^n_{\mathbb{C}}$ is still surjective.

(b) Show that if $z, w \in H^n_{\mathbb{C}}$ and $z \sim w$, but $z \neq w$, then $z_{n+1} = w_{n+1} = 0$.

(c) Show that the map $(z_1, \ldots, z_n, z_{n+1}) \mapsto (z_1, \ldots, z_n)$ defines a homeomorphism $H^n_{\mathbb{C}} \to B^{2n}$.

Remark: Question 2 shows that (a) & (b) imply that $\mathbb{P}^n(\mathbb{C}) = H^n_{\mathbb{C}} \cup_f \mathbb{P}^{n-1}(\mathbb{C})$, where f is the restriction of p_n to the complex equator $S^{2n-1} = \{z \in S^{2n+1} \mid z_{n+1} = 0\}$.

(7) Let $f\colon S^1 \to X$ be a continuous map. Show that the following are equivalent:

(a) f is nullhomotopic.

(b) There exists a continuous map $g\colon B^2 \to X$ such that $g|_{\partial B^2} = f$.

(8) Let X be a topological space and let $x, y, z, w \in X$. Let f, g, h be paths from x to y, y to z, and z to w, respectively. Show that the paths $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are homotopic relative endpoints. (Remark: You need to write down an explicit homotopy.)

(9) Using the isomorphism $\Phi\colon \mathbb{Z} \to \pi_1(S^1, 1)$ discussed in lectures, show that every group homomorphism $\phi\colon \mathbb{Z} \to \mathbb{Z}$ can be written as $\phi = \Phi^{-1} \circ f_* \circ \Phi$ for some map $f\colon (S^1, 1) \to (S^1, 1)$. (Hint: ϕ is determined by its action on a generator of $\mathbb{Z}$.)

(10) The Borsuk–Ulam theorem states that if $f\colon S^2 \to \mathbb{R}^2$ is a continuous map, then there is a point $x \in S^2$ such that $f(x) = f(-x)$.

Can something like this be true for the torus $T^2 = S^1 \times S^1$ in place of S^2? I.e. is it true that for every map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ there exists a point (z, w) such that $f(z, w) = f(-z, -w)$?