Number Theory Example Sheet 4
Michaelmas 2004

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(Questions marked with a $*$ are optional.)

(1) Assuming that $\pi = 3.1415926...$ correct to seven decimal places, prove that the first three convergents to π are:

$$\frac{22}{7}, \quad \frac{333}{106}, \quad \frac{355}{113}.$$

Verify that $|\pi - 355/113| < 10^{-6}$.

(2) Find the fundamental solutions of the Pell equations $x^2 - Ny^2 = 1$ for $N = 5$, 7, 11, 13, 17.

(3) Find two solutions in positive integers for each of the equations $x^2 - 21y^2 = 1$, $x^2 - 29y^2 = 1$.

(4) Prove that the number with continued fraction $[10, 10^{2!}, 10^{3!}, \ldots]$ is transcendental.

(5) Following the examples in class, use the continued fraction algorithm to factor the numbers: 9509, 13561, 8777.

(6) Let M, N be positive integers such that N is not a square, and $M \leq \sqrt{N}$. If x, y is a solution of the equation $x^2 - Ny^2 = M$, prove that x/y is a convergent of $\sqrt{N}$.

(7) Use Pollard's $p - 1$ method with $k = 840$ and $a = 2$ to try to factor $n = 53467$. Then try with $a = 3$.

(8∗) Prove that, if (x_n, y_n) for $n = 1, 2, ...$ is the sequence of positive solutions of the Pell equation $x^2 - Ny^2 = 1$ written in increasing values of x_n, y_n, then x_n and y_n satisfy a recurrence relation

$$u_{n+2} - 2au_{n+1} + u_n = 0$$

where a is a positive integer. Find a when $N = 7$.