Examples 4: minimal model theory

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(1) Let (X, B) be a klt pair. Assuming the log minimal model program in dimension $\dim X$, show that there exists a *terminal* pair (Y, D) and a projective birational morphism $f\colon Y \to X$ with $K_Y + D = f^*(K + B)$.

(2) Assume the minimal model program for $\mathbb{Q}$-factorial klt pairs of dimension d.

Explain how one can extend the minimal model program to klt pairs (X, B) where X is not necessarily $\mathbb{Q}$-factorial.

Show that, if $f\colon X \to Y$ is a birational contraction and the exceptional set of f contains a divisor, then the exceptional set is of pure codimension one, and it has at most $1 + \operatorname{rk}(\operatorname{Wei} X/\operatorname{Pic} X)$ irreducible components.

Show by example that sometimes it is necessary to flip divisorial contractions. [Hint: toric varieties]

Show that flips exist.

(3) Let $f\colon (X, B) \to Z$ be a 2-dimensional relative weak del Pezzo terminal pair. In other words:

1. X is a nonsingular surface and $\operatorname{mult}_x B < 1$ for all points $x \in X$, and
2. $-(K + B)$ is f-nef and f-big.

Let $\mathbf{M}$ be a mobile satuated b-divisor. Show that one of the following holds

1. $\mathbf{M}$ descends to X,
2. the linear system $|H^0(X, \mathbf{M})|$ is composed of an elliptic pencil and it has exactly one base point,
3. $-(K + B) \cdot \mathbf{M}_X = 1$.

[I am sure this is too hard; try for a while and then look at Ch 6 of *Flips for 3-folds and 4-folds*.]

(4) Prove Shokurov's finite generation conjecture for complete surfaces. In other words, let (as in the previous question) Z be an affine variety, $f : (X, B) \to Z$ be a 2-dimensional relative weak del Pezzo terminal pair, and $R = R(X, \mathbf{M}_\bullet)$ a Shokuorov algebra on X. Show that R is finitely generated.

Hint: use the previous question. Case (3) disappears after truncation and case (2) requires some work.