M3/4/5P21 - Algebraic Topology

Imperial College London
Lecturer: Professor Alessio Corti
Notes typeset by Edoardo Fenati and Tim Westwood

Spring Term 2014

These lecture notes are written to accompany the lecture course of Algebraic Topology in the Spring Term 2014 as lectured by Prof. Corti. They are taken from our own lecture notes of the course and so there may well be errors, typographical or otherwise. As a result use these notes with due care, and we would be grateful if you could email us any corrections or comments to tjw211@imperial.ac.uk or ef911@imperial.ac.uk.

We hope someone finds these notes useful!

Edoardo Fenati and Tim Westwood
January 2014

Contents

Chapter 1

Introduction

1.1 Recommended Reading

- Allen Hatcher, Algebraic Topology. Cambridge University Press.
Available at, `http://www.math.cornell.edu/~hatcher/AT/ATpage.html` This course will (hopefully) cover the content of chapters 1 and 2.
- Singer and Thorpe, Lecture notes on Elementary Topology and Geometry.

1.2 A Course Overview

This course will define *algebraic invariants of topological spaces*. This will be done by defining *functors*;

$$F : \underline{Top} \to \underline{Gp}$$

Where $\underline{Top}$ is the catergory of Topological Spaces and $\underline{Gp}$ is the category of Groups, most of the time these groups will be abelian, $\underline{Ab}$.

It is intuitive to think of a category as a 'thing' with objects and morphisms. For example, in $\underline{Top}$ the objects are topological spaces and the morphisms are the continuous maps. Similarly in $\underline{Gp}$, the objects are groups and the morphisms are group homomorphisms.

Moreover, funtors can be thought of as a thing that sends objects and morphisms to objects and morphisms in a consistent way.

It is useful to ask the following questions:
1) What kinds of topological spaces are we interested in? In this course we will consider CW complexes.
2)What sort of F? We will define two functors, $\pi_1, H_\star$.

1.3 Background from Point-Set Topology

There following is a useful proposition:

Proposition 1. *If X is a T_2 (Hausdorff) topological space and $K \subset X$ is a compact subset then K is closed.*

1.4 The Quotient Topology

Definition 1. *Let X be a topological space, $R \subset X \times X$ be a (set theoretic) equivalence relation. Furthermore let*

$$\pi : X \to {}^{X}\!/_{R} = Y$$

be the natural map. Then define the quotient topology on Y to be the topology such that

$$U \subset Y \text{ is open} \iff \pi^{-1}(U) \text{ is open in } X$$

The quotient topology is the 'biggest' topology that makes π continuous. It makes sense to consider the 'biggest' topology since the trivial topology is the 'smallest' topology.

1.4.1 Universal Property Characterising the Quotient Topology

If $f : X \to Z$ is continuous and constant on classes then there exists a unique map $g : Y \to Z$ such that $g \circ \pi = f$ is continuous. The existence and uniqueness of such a map follows results of set theory.

This result can also be considered in terms of the following diagram, where the result is that the diagram commutes.

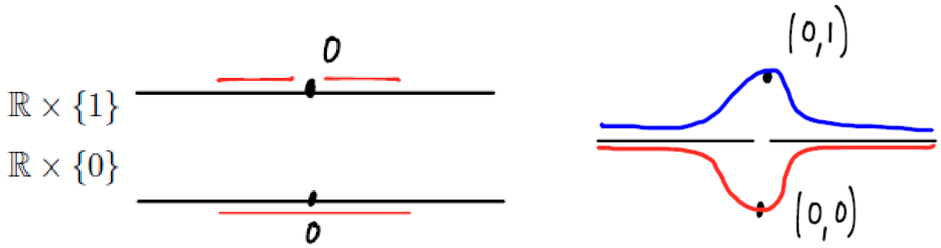
$$\begin{tikzcd}
X \arrow[dd, "\pi"'] \arrow[dr, "f"] & \\
& Z \\
Y \arrow[ur, dashed, "g = \pi \circ f"'] &
\end{tikzcd}$$

We now consider an example of non Hausdorff quotien.t
Example 1. Let $X = \mathbb{R} \times \{0, 1\}$, (a).
Define the equivalence classes of the relation R to be the sets,

$$\{(t, 0), (t, 1) \mid t \neq 0\}, \{(0, 0)\}, \{(0, 1)\}$$

Then ${}^{X}\!/_{R}$, is a line with the origin 'doubled up' and can be graphically represented as in the figure above (b). A small neighbourhood of $(0, 0)$ does not include $(0, 1)$ since the preimage must be open. Moreover any neighbourhoods of $(0, 0)$ and $(0, 1)$ must always intersect. Therefore ${}^{X}\!/_{R}$ is not Hausdorff!

This case is rather pathological however we will only consider T_2 spaces in this course. However it is not always easy to ensure that a quotient space is T_2.



(a) X (b) $X/_{R}$

A useful observation is the following. Suppose that $Y = X/_{R}$ is T_2 then it is clear that 1-element sets $\{y\} \subset Y$ are closed. Therefore $\pi^{-1}(y) = [x]$ is also closed in X. In other words if Y is T_2 then the classes of R must be closed in X.

Theorem 1. *Let $\pi : X \to Y$ be continuous, surjective TFAE:*

1. *π is a quotient map for R such that the classes are the fibres of π. Equivalently, Y has a quotient topology.*
2. *$U \subset Y$ open $\iff$ $\pi^{-1}(U)$ is open.*
3. *If for all topological spaces, Z, $f : Y \to Z$ is continuous $\iff$ $f \circ \pi$ is continuous.*

Theorem 2. *Let X, Y be topological spaces, $\pi : X \to Y$ be a surjective, continuous map. Further suppose X is compact, Y is Hausdorff. Then π is a quotient map.*
(Informally, π bijective $\implies$ π homeomorphism)

Proof. We want to prove that;

$$\pi^{-1}(U) \subset X \text{ open } \implies U \subset Y \text{ open}$$

To show this let;

$$K = X \backslash \pi^{-1}(U)$$

This is closed, since π^{-1} is assumed to be open. Since K is closed it is also compact (a closed subset of a compact set). This implies that $\pi(K) \subset Y$ is compact as π is continuous. By the assumption that Y is T_2 it follows that $\pi(K)$ is closed. And hence $U = Y \backslash \pi(K)$ is open. □

There is a slight subtly here. A map $f : X \to Y$ is said to be open if it maps open sets to open sets, however π may not necessarily by open. Consider the following example,

Example 2. *Let S^1=the circle =$\{z \in \mathbb{C} | |z| = 1\}$. Let $f : [0,1] \to S^1$ be the map such that $f(t) = exp(2\pi i t)$.*

Then this is a quotient map by the above theorem, $[0,1]$ compact (by Heine Borel), S^1 is Hausdorff, and f is continuous and surjective. Therefore f is a quotient map, however f is not open!

1.5 Construction of some Topological Spaces

We construct some standard topological spaces while introducing some stand methods for constructing topological spaces. Consider the following constructions;

(1) Let X be a topological space, $A \subset X$, and $f : A \to Y$ be a quotient map. We let

$$X \cup_f Y = X/_{R}$$

Where R is the relation that says that, for all $x_1, x_2 \in A$, $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

When $Y = \{pt\}$ then this is called *collapsing*, often denoted (rather confusingly) by X/A.

(2) Assuming the same set up as above, but let $f : A \to Y$ be any continuous map, (f is NOT surjective). Then;

$$X \cup_f Y = X \sqcup Y/R$$

Where R is the equivalence relation generated by $x \sim f(x)$ for $x \in A$. We now consider a few examples. Note $\simeq$ is used to mean 'is homeomorphic to'.

Example 3. Let $X = [0,1]$, $A = 0,1$ then $X/A \simeq S^1$. This is an example of collapsing.

$$\begin{tikzcd}
{[0,1]} \arrow[d, "\pi"'] \arrow[dr, "e^{2\pi i t}"] & \\
X/A \arrow[r, dashed, "\phi"'] & S^1
\end{tikzcd}$$

Proof.
There is an induced continuous map $\pi : X/A \to S^1$ by the universal property of quotients. Then ϕ is bijective, X/A is compact, S^1 is T_2 and so by theorem 2, ϕ is a homeomorphism. $\square$

This can also be taken as the definition of S^1.

Example 4. Let $S^2 = \{x \in \mathbb{R} | |x| = 1\}$ be the two dimensional sphere, $D^2 = \{y \in \mathbb{R}^2 | |y| \leq 1\}$ be the two dimensional disc and $\partial = \{y \in \mathbb{R}^2 | |y| = 1\} = S^1$, that is ∂ is the boundary of D^2. Then;

$$D^2/\partial \simeq S^2$$

Proof. Graphically the argument used is to cover the sphere with the two dimensional disc. In a similar way to 'putting a hat on', or 'putting a ball in a draw string bag',this homeomorphism can be seen graphically.

Formally we need to define a map f between the surfaces, such a function is ;

$$f(y) = \begin{cases} (2y, \sqrt{1-4y^2}) & \text{if } |y| \leq \frac{1}{2} \\ (\frac{\sqrt{1-(1-2r)^2}y}{|y|}, -2|y|+1) & \text{if } |y| \geq \frac{1}{2} \end{cases}$$

It is useful to consider the following diagram;

$$\begin{tikzcd}
D^2 \arrow[d, "\pi"'] \arrow[dr, "f(y)"] & \\
D^2/\partial \arrow[r, dashed, "\phi"'] & S^2
\end{tikzcd}$$

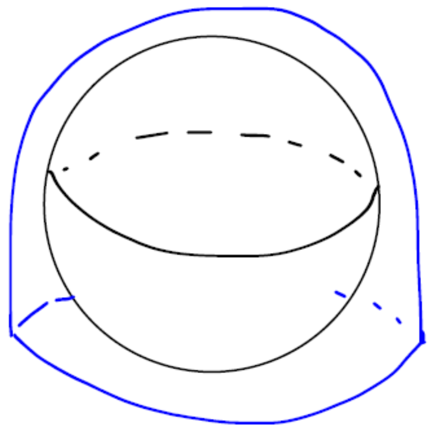


Figure 1.2: Sphere covered with the disc D^2

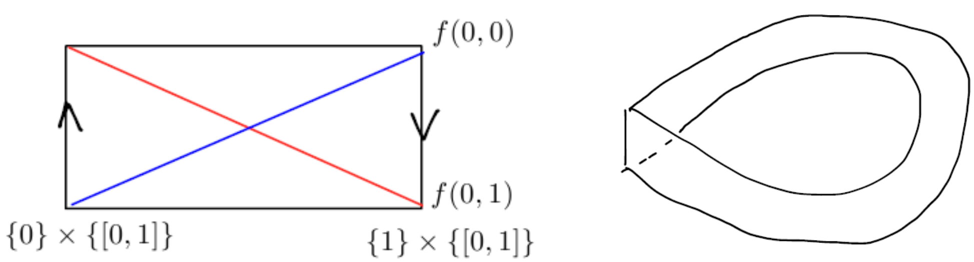
The existence of $\phi : {D^2}\big/{\partial} \to S^2$ follows from the universal property of quotients, the argument then follows identically to as above. $\square$

Intuitively the inside of the disc maps onto the sphere minus the south pole, the boundary is 'identified' with the south pole in the quotient.

Example 5. *Let M be the Mobius strip $[0,1]^2 \cup_f [0,1]$, $A = \{0\} \times [0,1] \sqcup \{1\} \times [0,1]$. Where $f : A \to [0,1]$ is a continuous map such that;*

$$f(0,t) = t, f(1,t) = 1 - t$$

Intuitively, one pair of opposite edges of the square are 'identified' with each other, however the top left hand corner is send to the bottom right corner (red), and similarly the bottom left hand corner is sent to the top right hand corner (blue).



(a) The effect of f on A

(b) The Mobius Strip

Example 6. We now consider the real projective plane. We begin by defining the relation $\sim$, such that $x \sim y \iff \exists \lambda \neq 0$ such that $x = \lambda y$. The it follows that;

$$\mathbb{RP}^2 = {\mathbb{R}^3 \backslash \{0\}}\big/{\sim}$$

However this is not the only descriptions of the real projective plane. Two others are;

- ${S^2}\big/{\sim}$, where $\sim$ is the relation such that the equivalence classes are $\{(-x, x)\}$.
- $D^2 \cup_f S^1$, $A = \partial D^2 = S^1$. Such that $f : S^1 \to S^1$, $f(z) = z^2$, $z \in \mathbb{C}, |z| = 1$. That is the antipodal points of the sphere are identified.

We claim that tall of these characterisations are equivalent. In order to show this, we begin by defining maps between D^2, S^2 and $\mathbb{R}^3$, that is;

$$D^2 \xrightarrow{g} S^2 \xrightarrow{h} \mathbb{R}^3 \backslash \{0\}$$

where g is the map that sends $y \mapsto (y, \sqrt{1-|y|})$ where $|y| \leq 1$, and h is the natural inclusion map. It is almost imediate that these maps are continuous (by the universal property), and bijective maps (by inspection). (The inverse of g is the map that sends $x \mapsto \frac{x}{|x|}$. These maps also respect the classes of the respective relations therefore;

$$D^2 \cup_f S^1 \xrightarrow{[g]} S^2 \big/ \sim \xrightarrow{[h]} \mathbb{RP}^2$$

In order to show that;

$$S^2 \big/ \sim \simeq \mathbb{RP}$$

It is possible to prove the following equalities;

$$[r] \circ [h] = \mathrm{id}_{S^2 \big/ \sim}$$

and

$$[h] \circ [r] = \mathrm{id}_{\mathbb{RP}^2}$$

This implies that $S^2 \big/ \sim \simeq \mathbb{RP}^2$.
However to prove that;

$$D^2 \cup_f S^1 \simeq S^2 \big/ \sim$$

we use theorem 2. Therefore we need to show that $S^2 \big/ \sim$ is T_2. A sketch proof of this is now given.

Proof. Let $x, y \in S^2$ have distinct images in $S^2 \big/ \sim$; this means that $x \neq \pm y$ in S^2. We need to construct disjoint neighbourhoods U of $\{x, -x\}$, and W of $\{y, -y\}$ such that $U = -U$ and $W = -W$. Taking $U = B(x, \varepsilon) \sqcap B(-x, \varepsilon)$, and $U = B(y, \varepsilon) \sqcap B(-y, \varepsilon)$ for $0 < \varepsilon$ sufficiently small will do (we used the notation $B(x, \varepsilon) = \{v \mid \|v - x\| < \varepsilon\}$). Observe that $\{x, -x\}, \{y, -y\} \in S^2 \big/ \sim$ we aim to construct disjoint neighbourhoods of these distinct points. □

1.6 CW Complexes

A CW Complex is a space built inductively, that is $X = \bigcup_{k=0}^{n} X^k$.
Let X^0 be a finite set then define;

$$X^k = \amalg_a^{N_k} D_a^k \cup_f X^{k-1}$$

Where, $f = \amalg_a f_a : \partial D_a^k = S_a^{k-1} \to X^{k-1}$, $D^k = \{x \in \mathbb{R}^k | |x| \leq 1\}$ and ∂D^k the boundary of D^k.

X^k is known as the kth skeleton of X. D_a^k are called the k dimensional cells of the CW complex.

All of the space we talk about in this course are homeomorphic (or homotopic) to CW complexes. Moreover the invariants, $(\pi_q(x, x_0), H_0 x)$ are computable on CW complexes.

Example 7. Consider

$$\mathbb{RP}^n = \mathbb{R}^{n+1}\backslash\{0\}\Big/\sim$$

where the relation is defined as $x \sim y$ if and only if $\exists\lambda \in \mathbb{R}^*$ such that $x = \lambda y$. Moreover this is equivalent to,

$$\mathbb{RP}^n = S^n\big/\sim$$

where $\sim$ is the antipodal equivalence. However $\mathbb{RP}^n$ can be considered as;

$$\mathbb{RP}^n = D^n \cup_f \mathbb{RP}^n = D^n \cup_f S^n\big/\sim$$

where $\sim$ is the antipodal equivalence, where f is the quotient map such that $f : \partial D^n = s^{n-1} \to \mathbb{RP}^n$.

In the last form it is clear that $\mathbb{RP}^n$ is exhibited as CW complex.
Example 8. We now consider the n dimensional complex projective space $\mathbb{CP}^n$.

$$\mathbb{CP}^n = \mathbb{C}^{n+1}\backslash\{0\}\Big/\sim$$

where $x \sim \lambda y$, $\lambda \in \mathbb{C}^*$.

$$\mathbb{CP}^n = S^{2n+1}\Big/\sim$$

Where $x \sim \lambda x$ when $\lambda \in \mathbb{C}, |\lambda| = 1$.
(Exericise: Why is this a CW complex?)
Example 9. We consider a surface with attachments at the boundary. See figure 1.4, 1.5 and 1.6.

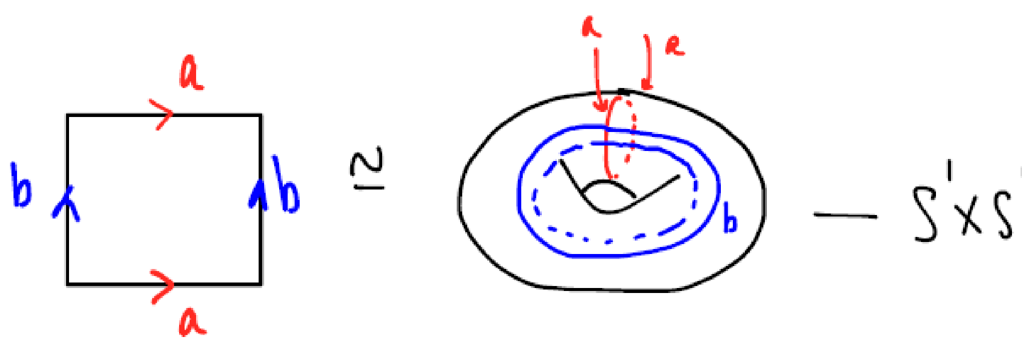


Figure 1.4: Constructing the Torus

1.6.1 The Euler Number of a CW complex

Definition 2. *The Euler number of CW complex X is defined to be;*

$$\begin{aligned} e(X) &= \sum_{k=0}^{n}(-1)^k\#\{k \textit{ dimensional cells that are added in at the } k \textit{ level}\} \\ &= \sum_{k=0}^{n}(-1)^k N_k \end{aligned}$$

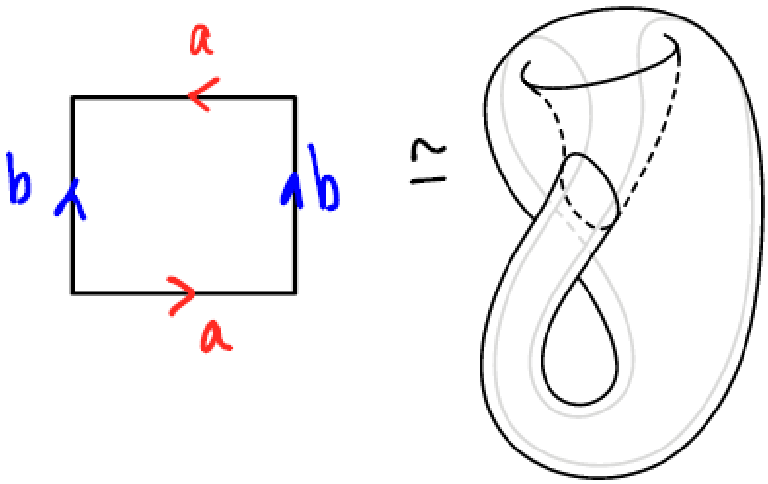


Figure 1.5: Constructing the Klein Bottle

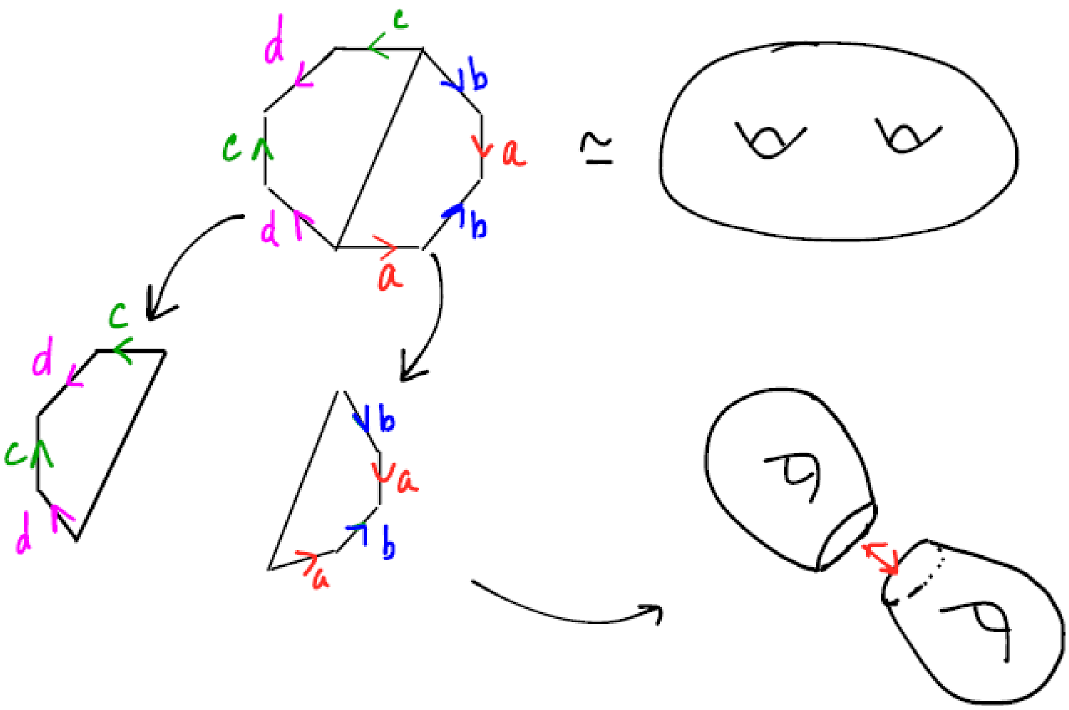


Figure 1.6: Constructing a Torus with two Holes

It is a fact that $e(X)$ us another topological invariant of a space X. Moreover, $e(X)$ is a homotopy invariant of X.
We now consider the calculation of Euler Numbers when S^2 is considered to be a CW complex.

Example 10. *We give some examples of when S^2 can be considered as a CW complex. Note in all of these cases $e(X) = 2$, it is indeed true that this is true for an arbitrary cellular decomposition of S^2.*

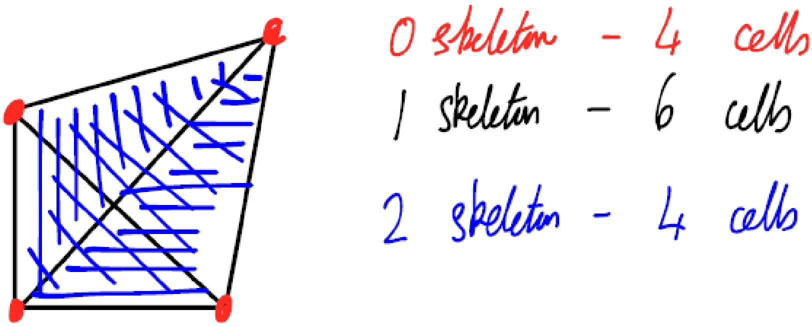


Figure 1.7: $e(X) = 4 - 6 + 4 = 2$

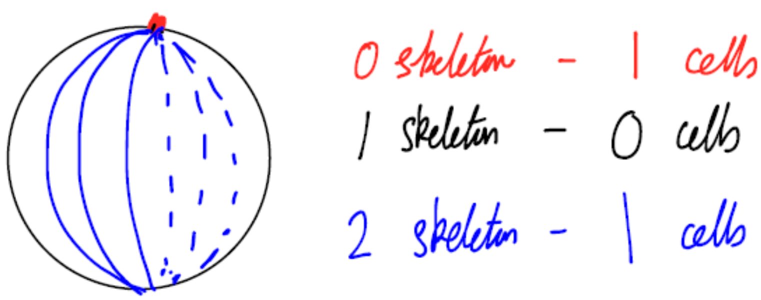


Figure 1.8: $e(X) = 1 - 0 + 1$

Figure 1.9: $e(X) = 2 - 1 + 1$

Chapter 2

Homotopy

2.1 Path Homotopy

Definition 3. *Let X be a topological space and let $a, b \in X$. A path from a to b in X is a continuous map $\gamma : I \to X$ (where $I := [0,1]$) with $\gamma(0) = a$ and $\gamma(1) = b$. A loop in X based at a is a path from a to a.*

Definition 4. *Let X be a topological space and $\alpha, \beta : I \to X$ be two paths from a to b. A homotopy from α to β is a continuous map $F : I \times I \to X$ such that:*

1. $F(t, 0) = \alpha(t)$ *and* $F(t, 1) = \beta(t) \quad \forall t \in I$
2. $F(0, s) = a$ *and* $F(1, s) = b \quad \forall s \in I$

In other words a homotopy is a path of paths with endpoints the paths α and β. When two paths α and β are related in this way by a homotopy F, they are said to be homotopic. The notation is $\alpha \sim \beta$

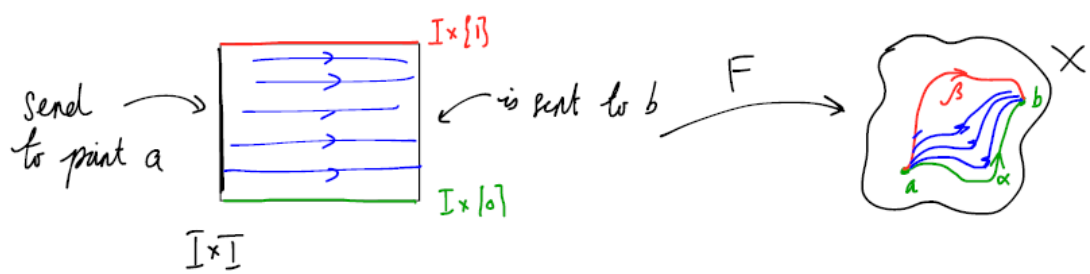


Figure 2.1: Graphical Representation of Homotopy of Paths

Definition 5. *Given two paths $\alpha, \beta : I \to X$ such that $\alpha(1) = \beta(0)$, there is a composition or product path $\alpha \circ \beta$ that traverses first α and then β, defined by the formula:*

$$\alpha \circ \beta(t) = \begin{cases} \alpha(2s) & 0 \leq t \leq 1/2 \\ \beta(2s-1) & 1/2 \leq t \leq 1 \end{cases}$$

Thus α and β are traversed twice as fast in order for $\alpha \circ \beta$ to be traversed in unit time.

Theorem 3. $\alpha_1 \sim \alpha_2, \beta_1 \sim \beta_2 \implies \alpha_1 \circ \beta_1 \sim \alpha_2 \circ \beta_2$

Figure 2.2: Composition of Paths

Proof. Suppose; Is a homotopies from α_1 to α_2 and from β_1 to β_2 respectively.

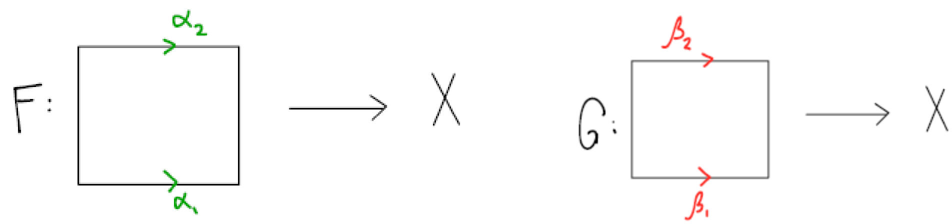
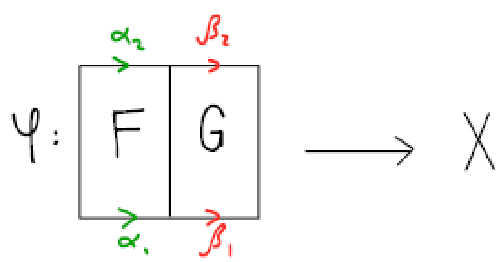


Figure 2.3: Composition of Paths

Now consider the following homotopy, Then this is a homotopy form $\alpha_1\beta_1$ to $\alpha_2\beta_2$.



Then;

$$\phi(t,s) = \begin{cases} F(2t,s) & \text{if } 0 \le t \le 1/2 \\ G(2t-1,s) & \text{if } 1/2 \le t \le 1 \end{cases}$$

Proves the theorem. ☐

2.1.1 A more general definition of Homotopy

We now formally state the definitions of homotopy, relative homotopy, homotopy equivalence and what it means to say that two spaces are homotopy equivalent.

Definition 6. *Let X, Y be topological spaces, let $f, g : X \to Y$ be continuous maps. f is said to be **homotopic** to g, denoted $f \sim g$, if and only if there exists a continuous function $F : X \times I \to Y$ such that;*

$$F(x,0) = f(x), F(x,1) = g(x) \qquad \forall x \in X$$

*F is called a **homotopy** from f to g.*

If $A \subset X$ is a topological subspace and $f|_A = g|_A$ then $f \sim g$ relative to A, often denoted 'rel A', if and only if $\exists$ a continuous $F : X \times I \to Y$ such that;

$$F(x, 0) = f(x), F(x, 1) = g(x) \qquad \forall x \in X$$

but also that F satisfies

$$F(x, t) = f(x) = g(x) \qquad \forall x \in A, \forall t \in I$$

Path homotopy defined previously was an example of a relative homotopy.

Definition 7. *$f : X \to Y$ is a **homotopy equivalence** if $\exists g : Y \to X$ s.t. $g \circ f \sim id_x, f \circ g \sim id_Y$.(where $\sim$ denotes homotopy)*
*X, Y are said to be **homotopy equivalent**, denoted $X \sim Y$ if $\exists f : X \to Y$ which is a a homotopy equivalence.*

Definition 8. *Let $i : A \hookrightarrow X$ be inclusion of a topological space.*
*A **retraction** from X to A is a continuous map $r : X \to A$ such that $r \circ i = id_A$.*

*A **retraction** r is a deformation retraction if $i \circ r \sim id_X$ rel A ($\implies$ r, i are homotopy equivalent)*

Example 11. Let

$$A = S^n \hookrightarrow \mathbb{R}^{n+1} \big/ \{0\} = X$$

Then define

$$r(x) = \frac{x}{||x||} \; \forall x \in X$$

is a retraction from X to A. Then compute $r \circ i$;

$$r \circ i(x) = \frac{x}{||x||} = i(x) = x, \; \forall x \in X$$

Since $||x|| = 1$ for all $x \in S^n$. Moreover, r is also a deformation retraction. In order to show this we need to define a homotopy such that $i \circ r \sim id_X$. That is define, $F : X \times I \to X$ such that

$$F(x, 0) = \frac{x}{||x||} \text{ and } F(x, 1) = x, \quad \forall x \in X$$

To do this define, for any $x \in X$,

$$F(x, t) = tx + (1 - t)\frac{x}{||x||}$$

Such a map is continuous and the start and end points agree with the definition of a homotopy. More over if $x \in A$ then $x = ||x||$ and so $F(X, t) = X$.

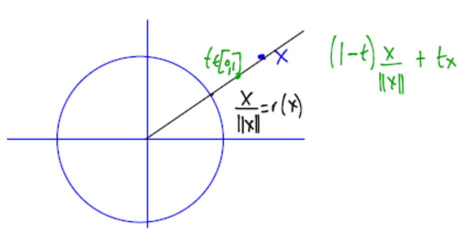


Figure 2.4: An example of a Deformation Retraction in $\mathbb{R}^2$

Definition 9. *A set X is called contractable if $f \sim \{pt\}$.*

Example 12. A convex set $C \subset \mathbb{R}^n$ is contractable.

Proof. let $c_0 \in C$, consider $i : \{c_0\} \hookrightarrow C$. Then $F(c,t) = tc + (1-t)c_0 \in C$, by definition of a convex set.

For $x \in C, t \in I$ is a homotopy from $r \equiv c_0$ to id_C relative to $\{c_0\} = A$. □

Example 13. *Letters of the alphabet - see Hatcher book.*

2.2 The Fundamental Group

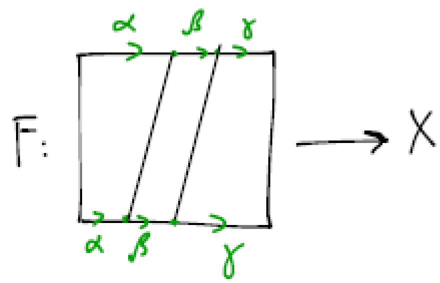
Definition 10. *Let X be a topological space and $a \in X$ then $\{\text{loops based at } a\}\big/\sim$ is a group called fundamental group π_1 where:*

1. $e : I \to X \quad e(t) = a \quad \forall t$
2. *if $\alpha : I \to X$ is a loop a t a then $\alpha^{-1} : I \to X$ is the loop $\alpha^{-1} = \alpha(1-t)$*

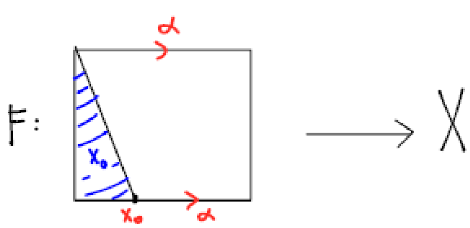
Theorem 4. $\pi_1(X, x_0) = \{\text{loops based at } a\}\big/\sim$ *is a group*

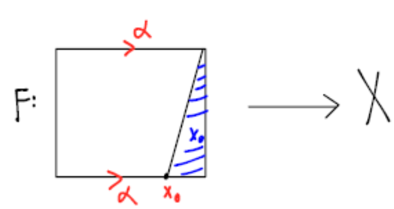
Proof. By the previous theorem on the composition of loops, it follows that the operation of composition defines an operation on $\pi_1(x_1, x_0)$. We prove this theorem by presenting graphically the homotopies required to prove the theorem.

For associativity, $(\alpha.\beta).\gamma \sim \alpha.(\beta.\gamma)$.

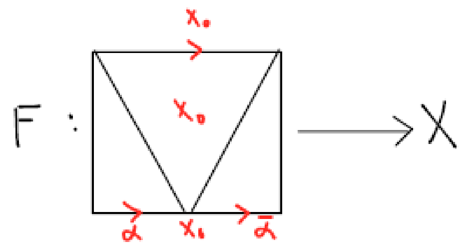


For the identity element, let $e : I \to X$ be the constant loop, $e(t) = x_0$ for all $t \in I$. We require that $e.\alpha \sim \alpha$, $\alpha.e \sim \alpha$ for all α.
Let $\alpha : I \to X$ be a path from a to b, define $\bar{\alpha} : I \to X$, $\bar{\alpha} = \alpha(1-t)$, be a path from b to a. Then





the homotopy can be defined as; □



Definition 11. *Let X be a topological space, it is path connected if $\forall x_0, x_1 \in X$ $\exists$ a path from x_0 to x_1*

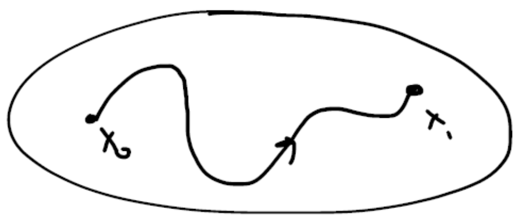


Figure 2.6: Path connectedness

Proposition 2. *X path connected $\implies$ X connected.*

Proof. Suppose for a contradiction that $X = U \amalg V$ where U and V are open, non-empty sets.

Then pick $x_0 \in U$, $x_1 \in V$. Let $\alpha : I \to X$ be a path from x_0 to x_1, which exists since X is assumed to be path connected. However it is possible to express I as;

$$I = \alpha^{-1}U \amalg \alpha^{-1}V$$

Which is a disjoint union since $0 \in \alpha^{-1}V$, $1 \in \alpha^{-1}U$, a contradiction because $I = [0, 1]$ is connected. □

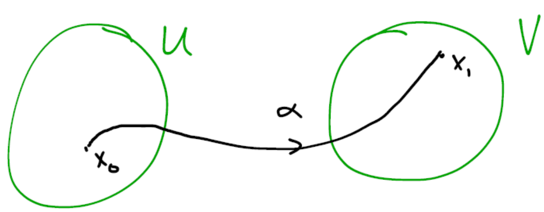


Figure 2.7: An illustration of the proof of proposition 2

Proposition 3. *Suppose X is a path connected topological space. Let $x_0, x_1 \in X$ then*

$$\pi_1(X, x_0) \simeq \pi_1(X, x_1)$$

Proof. Let $\alpha : I \to X$ be a path from x_0 to x_1.

$$\pi_1(X, x_1) \ni [\gamma] \xmapsto{\phi_\alpha} [\alpha] \cdot [\gamma] \cdot [\bar{\alpha}] \in \pi_1(X, x_0)$$

It is easy to check that ϕ_α is well defined, ϕ_α is a group homomorphism and $\phi_{\bar{\alpha}} = \phi_\alpha^{-1}$ □

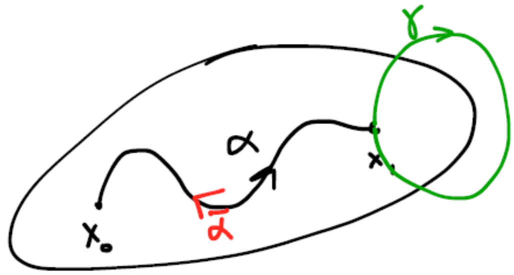


Figure 2.8: An illustration of the proof of proposition 3

2.3 Covering Maps

Definition 12. *A topological space X is locally connected if $\forall x \in V \subset X$, V open, $\exists\, x \in U \subset V$, U open and connected.*

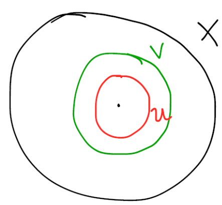


Figure 2.9: An illustration of the sets V, U, X in the definition of locally path connected

Definition 13. *A topological space X is locally path connected if $\forall x \in X$ and for all open neighbourhoods V of x, there exists an open neighbourhood $U \subset V$ of x such that U is path connected. i.e.$\forall\, x_1, x_2 \in U$ there exists path from x_1 to x_2 in V.*

Remarks:

1. It is essential that in the definition "for all open neighbourhoods V of x" is included, this ensures that there exist locally connected spaces that are not locally path connected.

2. All C.W complexes are locally connected and locally path connected.

3. Local connectedness and connectedness are not related to each other.

Definition 14. *Let X, $\tilde{X}$ be locally connected spaces and locally path connected spaces, $p : \tilde{X} \to X$ is a covering map if $\forall\, x \in X$, there exists an open set U, $x \in U$, such that*

$$p^{-1}(U) = \amalg_{y \in p^{-1}(x)} V_y$$

Such that $p|_{V_y} : V_y \to U$ is a homeomorphism $\forall\, y \in p^{-1}(x)$.

That is, $p : \tilde{X} \to X$ is a covering map if for all $x \in X$, and for all open neighbourhoods U of

X, the pre-image $p^{-1}(U)$ can be expressed as the disjoint union of open sets $V_y \subset \tilde{X}$. Where each V_y can be mapped homeomorphically onto U.

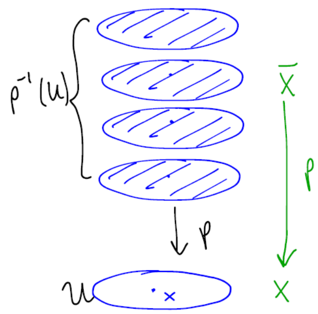


Figure 2.10: Illustration of the definition of a Covering Map

Example 14. *Define $p : \mathbb{R} \to S^1$, such that $p(t) = e^{2\pi i t}$. The p is an example of a covering map.*

2.4 Homotopy Lifting Theorems

The general problem of this section is the following; Given a covering map $p : \tilde{X} \to X$ and a function $f : Y \to X$ does there exist a lift of f to $\tilde{X}$ i.e a function $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$? And if such a lift exists is it unique? That is, does the following diagram commute?

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Theorem 5. *Suppose $\tilde{\alpha}, \tilde{\beta} : Y \to \tilde{X}$ such that $p \circ \tilde{\alpha} = p \circ \tilde{\beta} = f$ ($\tilde{X}$,X,Y locally connected, locally path connected, path connected). Suppose $\exists\, y_0 \in Y$ such that $\tilde{\alpha}(y_0) = \tilde{\beta}(y_0)$ then $\tilde{\alpha} = \tilde{\beta}$.*

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, shift left, "\tilde{\alpha}"] \arrow[ur, shift right, "\tilde{\beta}"'] \arrow[r, "f"'] & X \end{tikzcd}$$

Proof. Let $F = \{y \in Y \text{ s.t. } \tilde{\alpha}(y) = \tilde{\beta}(y)\}$. Note that $F \neq \emptyset$ since $y_0 \in F$. We claim that F is open and closed.

- We assume that $\tilde{X}$ is a Hausdorff space, this is equivalent to $\Delta = \{\tilde{x}, \tilde{x} | \tilde{x} \in \tilde{X}\}$ is closed.

Then consider;

$$F = (\tilde{\alpha} \times \tilde{\beta})^{-1}(\Delta) \qquad\qquad \tilde{\alpha} \times \tilde{\beta} : Y \to \tilde{X} \times \tilde{X}$$

The since $\tilde{\alpha} \times \tilde{\beta}$ is continuous and Δ is closed it follows that F is also closed.

• We suppose that $y_1 \in F$, that is that $\tilde{\alpha}(y_1) = \tilde{\beta}(y_1)$. Let $x_1 = f(y_1)$, then there exists $U \subset X$ open and connected, $x \in U$ s.t.;

$$p^{-1}(U) = \coprod_{\tilde{x} \in p^{-1}(x_1)} V_{\tilde{x}}$$

and $P|_{v_{\tilde{x}}} : V_{\tilde{x}} \to U$ is a homeomorphism for all $\tilde{x} \in P^{-1}(x_1)$. We will call such U an admissible open set in X.

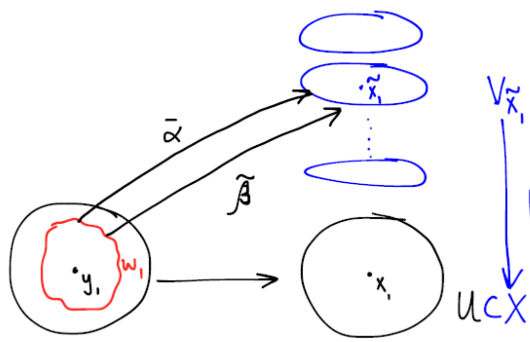


Figure 2.11: Proof of the first Homotopy Lifting theorem

Then there exists a neighbourhood $y \in W \subset Y$ s.t. $\tilde{\alpha}(w), \tilde{\beta}(w) \subset V_{\tilde{x}_1}$. BU then it is clear that;

$$\alpha|_w = (p|_{v_{\tilde{x}_1}})^{-1} \circ f|_w \text{ and } \beta|_w = (p|_{v_{\tilde{x}_1}})^{-1} \circ f|_w$$

That is;

$$\tilde{\alpha}|_w = \tilde{\beta}|_w$$

which implies that F is open.

Since Y is connected it follows from point set topology that the only open and closed sets are the empty set or the entire space. Since F is not empty it follows that $F = Y$ as required. $\square$

Theorem 6. *Let $\tilde{X}$, X, Y be locally connected, locally path connected and path connected topological spaces. Also suppose that Y is compact. Suppose that given a commutative diagram as fikkiws:*

$$\begin{tikzcd}
Y \arrow[r, "\tilde{f}"] \arrow[d, "i"'] & \tilde{X} \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & X
\end{tikzcd}$$

where $F \circ i = p \circ \tilde{f}$ then there exists $\tilde{F} : Y \times I \to \tilde{X}$ such that:

1. $p \circ \tilde{F} = F$
2. $\tilde{f} = \tilde{F} \circ i$

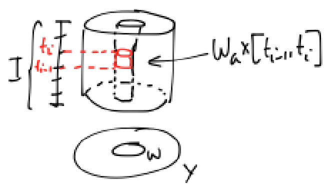
Proof. A simple compactness (of I and Y) argument shows that;

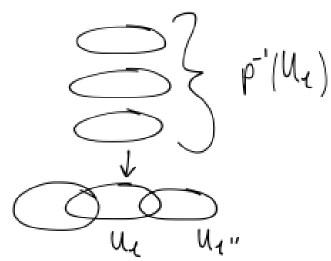
- There exists a covering $\{U_\ell\}$ of X by admissible open sets $U_\ell \subset X$.

- A covering $\{W_a\}$ of Y and a partition $0 = t_0 < t_1 < ... < t_m = 1$ of $I = [0, 1]$, such that;

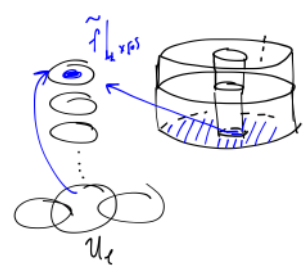
  $$F(W_\ell \times [t_{i-1}, t_i]) \subset U_\ell$$

  for all $i = 1, ...m$.





We can then construct $\tilde{F}$ on $Y \times [t_0, t_1]$, as in the picture.



□

Theorem 7. $\pi_1(S^1, 1) \simeq \mathbb{Z}$

Proof.

$$\Theta : \mathbb{Z} \to \pi_1(S^1, 1)$$

i.e $k \mapsto (t \xmapsto{\gamma_k} e^{2\pi ikt})$ Θ is surjective. Let $\alpha : [0, 1] \to S^1$ be a loop at 1.
Let $p(u) = e^{2\pi iu}$ be the lifting map. Now we apply theorem (6) with $Y = \{\text{pt}\}$ hence we have:

$$\tilde{\alpha} : [0, 1] \to \mathbb{R} \quad p \circ \tilde{\alpha} = \alpha$$

such that $\tilde{\alpha}(0) = 0$. Also observe that,

$$p\tilde{\alpha}(1) = \alpha(1) = 1 \implies \tilde{\alpha}(1) = k \in \mathbb{Z} \subset \mathbb{R}$$

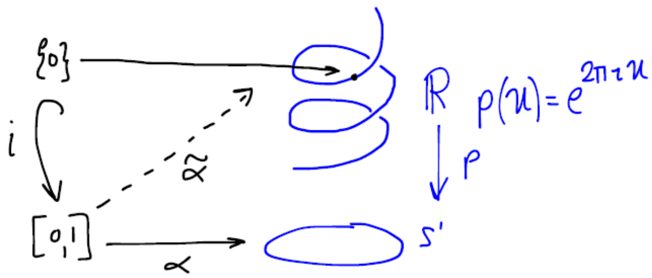


Figure 2.12: Proof of $\pi_1(S^1, 1) \simeq \mathbb{Z}$

We plan to show $[\alpha] = \Theta(k)$. $\tilde{\alpha}$ is any arbitrary path from 0 to k in $\mathbb{R}$. $\tilde{\alpha}$ is homotopic relative $\{0, 1\} \subset I$ to the path:

$$[0,1] \ni t \xrightarrow{\tilde{\gamma}_k} kt \in \mathbb{R}$$

We have that $\tilde{\alpha} \sim \tilde{\gamma}_k$. We define the homotopy F:

$$F : t, s \mapsto (1-s)\tilde{\alpha}(t) + kst$$

$$\begin{cases} F(t,0) & = \tilde{\alpha}(t) \\ F(t,1) & = \tilde{\gamma}_k(t) = kt \\ F(0,s) & = 0 \\ F(1,s) & = k \end{cases}$$

Now $p \circ F : I \times I \to S^1$ is a homotopy from α to γ_k, observe that the following hold;

$$\begin{cases} p \circ F(t,0) & = p \circ \tilde{\alpha}(t) = \alpha(t) \\ p \circ F(t,1) & = p \circ \tilde{\gamma}_k(t) = \gamma_k(t) \\ p \circ F(0,s) & = p(0) = 1 \\ p \circ F(1,s) & = p(k) = 1 \end{cases}$$

Therefore $\alpha \sim \gamma_k \implies [\alpha] = \Theta(k)$. Θ is injective. Supose $\Theta(k) = e \in \pi_1(S^1, 1)$. This means that $e \sim \gamma_k$ as a path i.e $\exists F : I \times I \to S^1$ such that:

$$\begin{cases} F(t,0) = 1 \\ F(t,1) = e^{2\pi i k t} \\ F(0,s) = F(1,s) = 1 \end{cases}$$

By theorem 6 $\exists \tilde{F} : I \times I \to \mathbb{R}$ lifting F such that:

$$\begin{cases} \tilde{F}(t,0) = 0 \\ \tilde{F}(0,s) = 0 \\ \tilde{F}(1,s) = 0 \end{cases}$$

but $p \circ \tilde{F}(t,1) = \gamma_k(t) = e^{2\pi i k t}$ so $\tilde{F}(t,1)$ is a lift of γ_k starting at $\tilde{F}(0,1) = 0$ but $t \mapsto kt$ is another such lift so by theorem (5) $\tilde{F}(t,1) = kt$ for all t. Hence we have $0 = \tilde{F}(0,1) = k$ $\square$

2.5 Functoriality of π_1

We now introduce the concept of the homotopy category of pointed topological spaces. Where the objects are pairs (X, x), where X is a path-connected topological space and $x \in X$.

Definition 15. *A pointed continuous map, $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y$.*

This allows for the following definition of a pointed homotopy.

Definition 16. *A pointed homotopy $f \sim g$ is a homotopy rel $\{x\}$. (recall this is a homotopy which when restricted to, the subset, $\{x\}$ is the identity map)*

Similarly to before, such a homotopy allows for the definition of an equivalence relation.

Definition 17. *A pointed homotopy equivalence, is a pointed continuous map f such that there exists a pointed continuous map $g : (Y, y) \to (X, x)$ such that;*

$$f \circ g \sim id_Y \; rel\{y\}, \quad g \circ f \sim id_X \; rel\{x\}$$

Moreover, $(X, x), (Y, y)$ are pointed homotopy equivalent if there exists a pointed homotopy equivalence between them.

In order for this to be a category, we need to define the morphisms that act on the objects. Define the morphisms to be the quotient of;

$$Mor((X, x), (Y, y)) = \{f : (X, x) \to (Y, y), \text{a p'td cont.map}\} \Big/ \{\text{p'td homotopy}\}$$

The following theorem has important implications.
Theorem 8. *Consider the following,*

$$\pi_1 : \{\textit{category of p'td topological spaces}\} \to \{\textit{category of groups}\}$$

Then this is a functor.

This has the following implications;

1. If $f \in Mor((X, x), (Y, y))$, then this induces a group homomorphism;

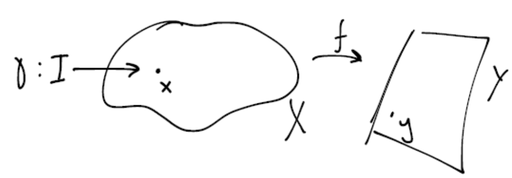
$$f_\star \pi_1(X, x) \to \pi_1(Y, y)$$

2. $(f \circ g)_\star = f_\star \circ g_\star$.

This follows from the following, let $f : (X, x) \to (Y, y)$ be a continuous map and let $\gamma : I \to X$ be a loop at x. Then define;

$$f_\star \gamma = f \circ \gamma$$

This is a loop at $f(x) = y$.



Corollary 1. *If (X,x) and (Y,y) have the same p'ted homotopy type then;*

$$\pi_1(X,x) \simeq \pi_2(Y,y)$$

CAUTION: This isomorphism is NOT canonical, it depends on the choice of homotopy equivalence.

2.6 Applications of the fundamental group

Theorem 9 (Fundamental theorem of Algebra)**.** *Every polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ $(a_i \in \mathbb{C})$ has at least 1 root.*

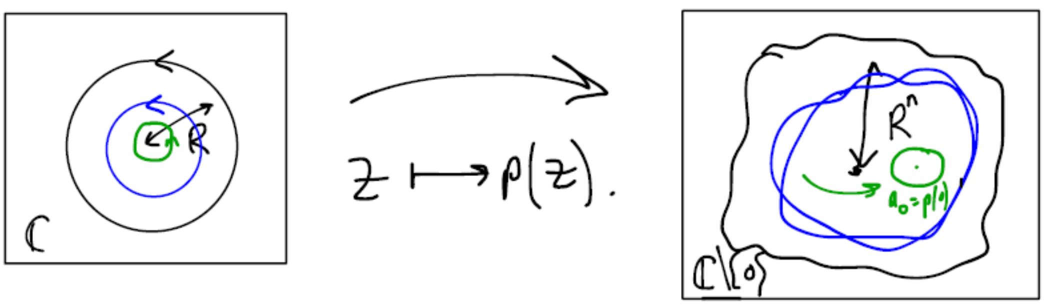


Figure 2.13: The proof of the FTA

Proof. Suppose for a contradiction that $p(z) \neq 0\ \forall z \in \mathbb{C}$.Let $r \geq 0$ be a real number and consider

$$t \mapsto f_r(t) = \frac{P(re^{2\pi i t})/P(t)}{|P(re^{2\pi i t})/P(t)|} \in S^1 \subset \mathbb{C}$$

This is a loop on S^1 based at 1.
$[0,1]\times[0,r_0] \ni t, r \mapsto f_r(t)$ as I move r this is a homotopy of loops. Fix $r = r_0 \geq$ max$\{|a_1|, |a_2|, \ldots, |a_n|, 1\}$.
If $|z| = r_0$ we have:

$$|z|^n = r_0 \geq r_0 r_0^{n-1} > (|a_1| r_0^{n-1} + |a_2| r_0^{n-2} + \cdots + |a_n|) \geq |a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n|$$

Now consider:

$$P_s(z) = z^n + s(a_1 z^{n-1} + \cdots + a_n)$$

This polynomial has no roots on $|z| = r_0$, for all $s \in [0,1]$ get

$$I \times I \ni (t,s) \to f_{r_0}(t,s) \in S^1$$

is a homotopy of loops on S^1. We get a homotopy from $t \to f_{r_0}(t)$ to the loop $\gamma_n : t \to e^{2\pi i n t}$ which is a contraddiction. □

Theorem 10 (Brouwer fixed point theorem)**.** $h : D^2 \to D^2$ *continuous* $\implies \exists x \in D^2$ *such that* $h(x) = x$

Proof. Suppose for a contradiction that $h(x) \neq x\ \forall x \in D^2$. Define $r : D^2 \to S^1$ as in the picture, r is a retraction of $i : S^1 \hookrightarrow D^2$ (if $x \in S^1, r(x) = x$). There can be no such retraction. It would

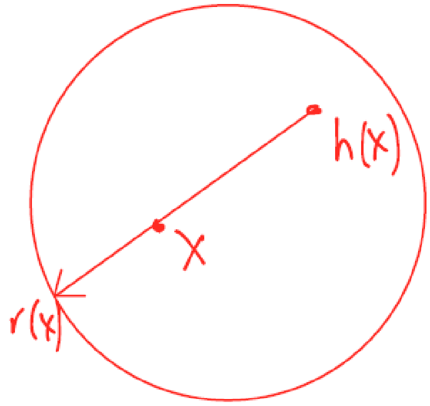


Figure 2.14: Construction of the retraction

contradict the functoriality of π_1

$$S^1 \overset{i}{\hookrightarrow} D^2 \xrightarrow{r} S^1 \qquad r \circ i = \text{id}_{S^1}$$

$$\underbrace{\pi_1(S^1)}_{\simeq \mathbb{Z}} \overset{i_*}{\hookrightarrow} \underbrace{\pi_1(D^2)}_{\simeq 0} \xrightarrow{r_*} \underbrace{\pi_1(S^1)}_{\simeq \mathbb{Z}}$$

$r_* \circ i_* = \text{id}_{S^1}$
$r_* \circ i_* = (r * i)_* = \text{id}_* = \text{id}$
We have a contradiction. □

Theorem 11 (Borsuk-Ulam)**.** *Let $f : S^2 \to \mathbb{R}^2$ be a continuous function then $\exists x \in S^2$ such that $f(x) = f(-x)$*

Proof. Suppose for a contradiction that $f(x) \neq f(x)\ \forall x \in S^2$. Define:

$$g : S^2 \to S^1 \qquad g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Consider the loop:

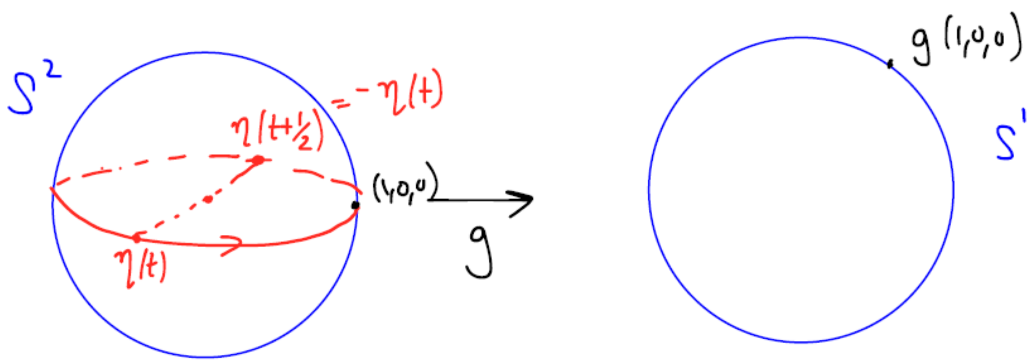


Figure 2.15: Proof of Borsuk-Ulam

$$t \to \eta(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

and the loop:

$$\gamma = g \circ \eta$$

The key point is to argue that γ is non-trivial. Note that $g(-x) = g(x)$ this implies:

$$\gamma(t + \frac{1}{2}) = -\gamma(t) \qquad t \in [0, 1]$$

Now lift γ to $\tilde{\gamma} : [0, 1] \to \mathbb{R}$, hence

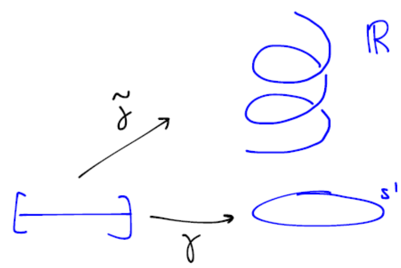


Figure 2.16: Lift of γ

$$\tilde{\gamma}(t + \frac{1}{2}) = \tilde{\gamma}(\frac{1}{2}) + \frac{k}{2} \qquad k \in \mathbb{Z} \quad \text{odd}$$

Independent of t by continuity therefore we have:

$$\tilde{\gamma}(1) = \tilde{\gamma}(\frac{1}{2}) + \frac{k}{2} = \tilde{\gamma}(0) + k \qquad k \in \mathbb{Z} \quad \text{odd}$$

therefore γ loops around an odd number of times $\implies$ $\gamma \neq 0$ in $\pi_1(S^1, g(1, 0, 0))$. This contradicts the functoriality of π_1 as $\gamma - g_* \eta$ and $\eta \sim 0$ on S^2 $\square$

Theorem 12. $\pi_1(S^2, *) = 0$

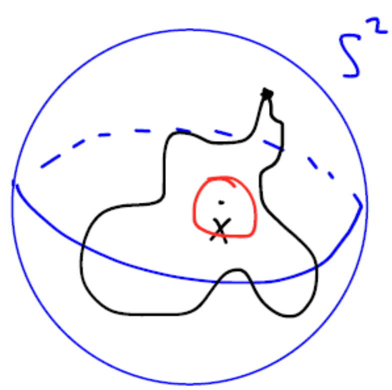


Figure 2.17: Proof of $Pi_1(S^2, \star) = (0)$

2.7 Free Groups with Amalgamation

Definition 18. *Suppose we are given a* ***diagram of Groups****, i.e.* G_1, G_2 *are groups and* H *is a subgroup of* G_1 *and* G_2*, together with the following diagram,*

$$
\begin{tikzcd}
G_1 & \\
H \arrow[u, "i_1"] \arrow[r, "i_2"] & G_2
\end{tikzcd}
$$

*The **categorical pushout** of this diagram is a commutative diagram;*

$$
\begin{tikzcd}
G_1 \arrow[r, "j_1"] & G_1 \underset{H}{*} G_2 \\
H \arrow[u, "i_1"] \arrow[r, "i_2"] & G_2 \arrow[u, "j_2"]
\end{tikzcd}
$$

*This diagram has the **universal property** of $j_2 \circ i_2 = j_1 \circ i_1$.*
This property is equivalent to;

$$
\begin{tikzcd}
G_1 \arrow[r, "p_1"] & G \\
H \arrow[u, "i_1"] \arrow[r, "i_2"] & G_2 \arrow[u, "p_2"]
\end{tikzcd}
$$

Such that $p_2 \circ i_2 = p_1 \circ i_1$, then there exists a unique $q : G_1 \underset{H}{*} G_2 \to G$, which makes everything commute. That is;

$$
\begin{tikzcd}
 & G_1 \arrow[r, "i_1"] \arrow[ddr, "p_1"] & G_1 \underset{H}{*} G_2 \arrow[dd, "\exists!\, q"] \\
H \arrow[ur, "i_1"] \arrow[dr, "i_2"] & & \\
 & G_2 \arrow[r, "p_2"] \arrow[uur, "j_2"] & G
\end{tikzcd}
$$

*Also $G_1 \underset{H}{*} G_2$ is called the **Free Product** of G_1 and G_2 with **amalgamation** of H.*

It is also worth noting that the push-out of groups is most similar to the direct product of commutative rings.

Theorem 13. *Push-outs of groups exist.*

Instead of giving a proof, an informal discussion is given instead.

1. If $H = \{e\}$ then $G = G_1 \underset{H}{*} G_2$ is the free product and G is the group of words. That is every element $g \in G$ can be expressed as $g = A_1 A_2 \dots A_r$ where $A_i \in G_1$ or G_2. Moreover, a word is said to be reduced if no consecutive ”letters” are in G_1 and G_2. Therefore G is the

set of reduced words.

2. $G_1 *_H G_2 = {G_1 * G_2}/{N}$ where $N \trianglelefteq G_1 * G_2$ is the normal subgroup generated by $\{i_1(h), h_2(h^{-1}) | h \in H\}$. It is really easy to see that ${G_1 * G_2}/{N}$ satisfies the required universal property to be the pushout.

3. Suppose you have G_1, G_2 & H given in terms of presentations:

$$G_1 = \langle u_1, \dots, u_k | r_1, \dots, r_l \rangle$$

$$G_2 = \langle v_1, \dots, v_m | s_1, \dots, s_n \rangle$$

$$H = \langle w_1, \dots, w_p | t_1, \dots, t_q \rangle$$

$$G_1 *_H G_2 = \langle u_1, \dots, u_k, v_1, \dots, v_m | r_1, \dots, r_l, s_1, \dots, s_n, i_1(w_1) i_2(w_1^{-1}), \dots, i_1(w_p) i_2(w_p^{-1}) \rangle$$

Example 15. Consider,

$$G = \mathbb{Z}/2 * \mathbb{Z}/2$$

Let A, B be the non identity elements in $G_1 = \mathbb{Z}/2$ and $G_2 = \mathbb{Z}/2$, then G is the set of words with letters A, B. E.g. a few examples of words are $A, B, AB, ABA, BABA, \dots$ etc.
It is possible to define a homomorphism;

$$\phi : \mathbb{Z}/2 * \mathbb{Z}/2 \to \mathbb{Z}/2$$

Such that, $\phi(\text{reduced word}) = 0$ if the length of the word is even and $\phi(\text{reduced word}) = 1$ if the length of the word is odd.
In fact,

$$\mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \ltimes \mathbb{Z} = \left\langle A, \langle AB \rangle =: C | A^2 = e, ACA = C^{-1} \right\rangle$$

This can be considered as the "infinite" dihedral group denoted by D_∞.

Similar arguments can be used to study $\mathbb{Z}/2 * \mathbb{Z}/3$, this is done in the following example.

Example 16. Let

$$\Gamma = PSL_2(\mathbb{Z}) = {SL_2(\mathbb{Z})}\Big/{\pm Id}$$

This is the group of transformations of $h^+ = \{z \in \mathbb{C} | Im(Z) > 0\}$ of the form,

$$z \mapsto \frac{az + d}{cz + d}, \quad \text{where} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad ad - bc \neq 0$$

Consider the following two specific rational functions of this form. Let,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{then } z \mapsto -\frac{1}{z}$$

And similarly;

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{then } z \mapsto z + 1$$

Therefore, after some effort it is possible to present Γ as;

$$\Gamma = \left\langle S, T | S^2 = \mathbb{1}, (ST)^3 = \mathbb{1} \right\rangle \implies \Gamma \simeq \mathbb{Z}/2 * \mathbb{Z}/3$$

2.7.1 Free Products and Presentations

Another way of determining the free product of groups is by considering the presentations. Suppose that G_1, G_2, H are given in terms of the presentations, where $u_1, \dots, u_k \in G_1$ are elements and $r_1, \dots, r_l$ are relations on the elements of G_1 etc...

$$G_1 = \left\langle u_1, \dots, u_k \middle| r_1, \dots, r_l \right\rangle, G_2 = \left\langle v_1, \dots, v_m \middle| s_1, \dots, s_n \right\rangle, H = \left\langle w_1, \dots, w_p \middle| t_1, \dots, t_q \right\rangle$$

Then;

$$G_1 \underset{H}{*} G_2 = \left\langle u_1, \dots, u_k, v_1, \dots, v_m \middle| r_1, .., r_l, s_1, \dots, s_n, i_1(w_1)i_2(w_1^{-1}), \dots, i_1(w_p)i_2(w_p^{-1}) \right\rangle$$

Example 17. We demonstrate the above idea by considering the free product of;

$$\sigma_3 \underset{\mathbb{Z}/2}{*} \mathbb{Z}/2$$

Firstly recall that σ_3 can be presented as $\sigma_3 = \langle a, b | a^3, b^3, baba \rangle$, and $\mathbb{Z}/4$ can be presented as $\mathbb{Z}/4 = \langle c | c^4 \rangle$. Similarly, $\mathbb{Z}/2$ has presentation, $\mathbb{Z}/2 = \langle d | d^2 \rangle$.
Then there exists two inclusion maps;

1. $i_1 : \mathbb{Z}/2 \hookrightarrow \sigma_3$ such that $i_1(d) = b$.

2. And, $i_2 : \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ such that $i_2(d) = c^2$.

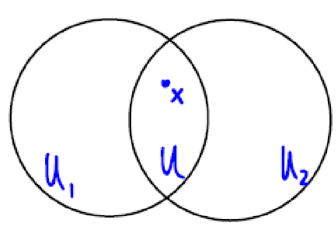
Therefore the required free product admits the following presentation;

$$\sigma_3 \underset{\mathbb{Z}/2}{*} \mathbb{Z}/2 = \left\langle a, b, c \middle| a^3, b^2, baba, c^4, bc^2 (= i_1(d) i_2(d^{-1})) \right\rangle$$

2.8 Seifert - van Kampen

Theorem 14. *Suppose $X = U_1 \cup U_2$, U_1 and U_2 are open, path connected. Let $U = U_1 \cap U_2$. For $x \in U$, then;*

$$\pi_1(X, x) = \pi_1(U_1, x) \underset{\pi_1(U,x)}{*} \pi_1(U_2, x)$$



This can be interpreted as the as the red part of figure 14, the 'pushout' is implied by the black part making the diagram commute. However this theorem requires some explanations which we will now give. A proof of this theorem is not given in this course, but one is easily found in the literature.

$$
\begin{tikzcd}
\pi_1(U_1, x) \arrow[r, "j_{1\star}"] & \pi_1(X, x) \\
\pi_1(U, x) \arrow[u, "i_{1\star}"] \arrow[r, "i_{2\star}"] & \pi_1(U_2, x) \arrow[u, "j_{2\star}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
U_1 \arrow[r, hook, "j_{1\star}"] & X \\
U \arrow[u, hook, "i_{1\star}"] \arrow[r, hook, "i_{2\star}"] & U_2 \arrow[u, hook, "j_{2\star}"']
\end{tikzcd}
$$

Figure 2.18: Pushout Commutative Diagrams

2.8.1 Applications of Seifert van-Kampen to computing the Fundamental Group

We now introduce the notation for the join of pointed topological spaces n(X_1, x_1) and (X_2, x_2);

$$X_1 \vee X_2 = X_1 \coprod X_2 \Big/ x_1 \sim x_2$$

An example of this is the following;
Example 18. *We will show that;*

$$\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$$

Proof. Observe that by the above definition of "$\vee$", $S^1 \vee S^1$ can be graphically represented as We aim to show this result using Seifert -van Kampen. Therefore let; Since U_1, U_2 deformation

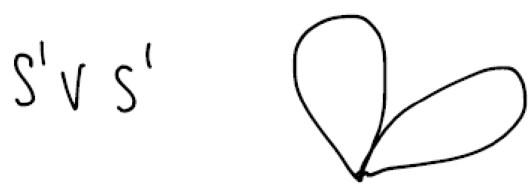


Figure 2.19: $S^1 \vee S^1$

Figure 2.20: The definition of U_1, U_2 and $U = U_1 \cap U_2$

retracts to S^1, and U deformation retracts to $\{x\}$, then Seifert-van Kampen implies that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$ $\square$

Similarly,
Example 19.

$$\pi_1(\overset{n}{\vee} S^1) = \mathbb{Z}^{*n}$$

Proposition 4. *Consider the standard 1-pt compactification of $\mathbb{R}^3 \subset S^3$. Then, the fundamental group is unaffected i.e.*

$$\pi_1(\mathbb{R}^3 \backslash K) = \pi_1(S^3 \backslash K)$$

Proof. This is left as an exercise and follows from the Seifert van-Kampen theory. $\square$

Example 20. Let $K = S^1 \subset \mathbb{R}^3$ then

$$\pi_1(\mathbb{R}^3 \backslash K) \simeq \mathbb{Z}$$

Proof. Observe that $S^3 \backslash K$ deformation retracts to $S^2 \cup$ Bar. To see this, observe that all points outside the sphere can be mapped to the surface of the sphere and any points within the sphere can be mapped to either the bar or the surface of the sphere. Now $S^2 \cup$ Bar deformation retracts to $S^2 \vee S^1$, therefore we have:

$$\pi_1(\mathbb{R}^3 \backslash K) \simeq \pi_1(S^2 \vee S^1) \simeq \mathbb{Z}$$

This proof can be seen graphically in the figure below. $\square$

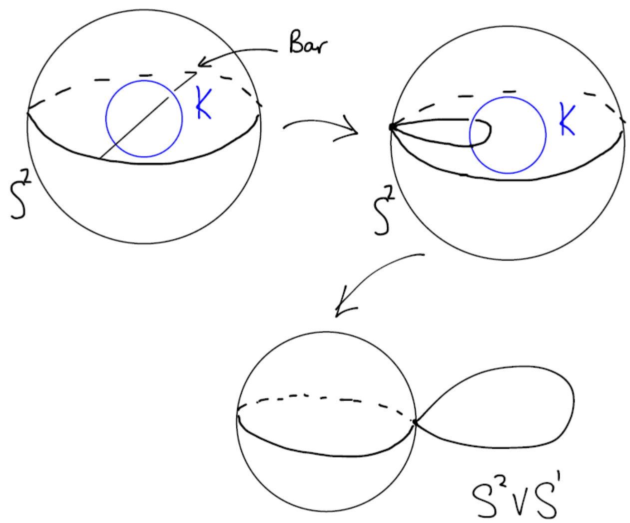


Figure 2.21: The deformation retraction of $\S^2 \backslash K$

Example 21. Let A and B be disjoint circles

$$\pi_1(\mathbb{R}^3 \backslash (A \cup B)) \simeq \mathbb{Z} * \mathbb{Z}$$

Proof. Observe that $\pi_1(\mathbb{R}^3 \backslash A \cup B)$ deformation retracts on $S^2 \vee S^1 \vee S^2 \vee S^1$ therefore we have:

$$\pi_1(\mathbb{R}^3 \backslash K) \simeq \pi_1(S^2 \vee S^1 \vee S^2 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$$

$\square$

Example 22. Let A and B be linked circles; Then

$$\pi_1(\mathbb{R}^3 \backslash A \cup B) \simeq \mathbb{Z} \times \mathbb{Z}$$

Proof. Using the above proposition above it is sufficient to consider $\pi_1(S^3 \backslash (A \cup B))$. Observe that S^3 can be expressed using the Heegaard splitting as;

$$S^3 = S^1 \times D^2 \bigcup_{S^1 \times S^1} S^1 \times D^2$$

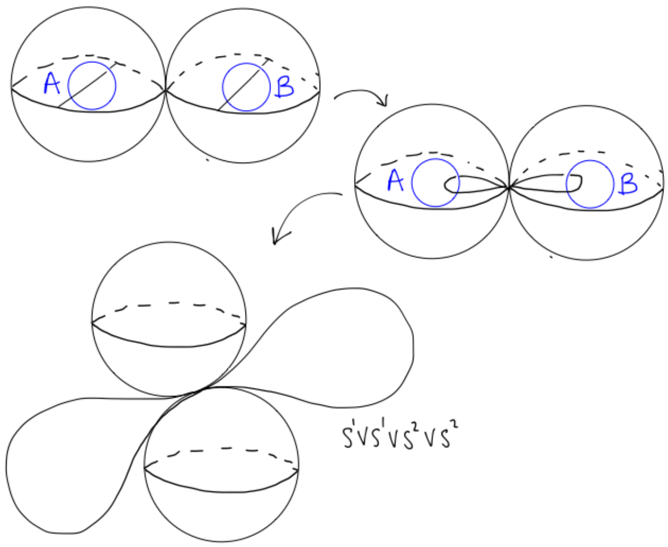
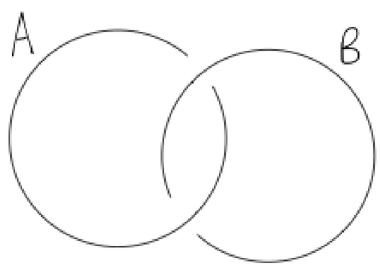


Figure 2.22: Deformations retractions in $\pi(\mathbb{R}^3\backslash K)$



Then $S^3\backslash(A\cup B)$ deformation retracts on the torus $S^1\times S^1$ therefore:

$$\pi_1(\mathbb{R}^3\backslash A\cup B)\simeq\pi_1(S^1\times S^1)\simeq\mathbb{Z}\times\mathbb{Z}$$

$\square$

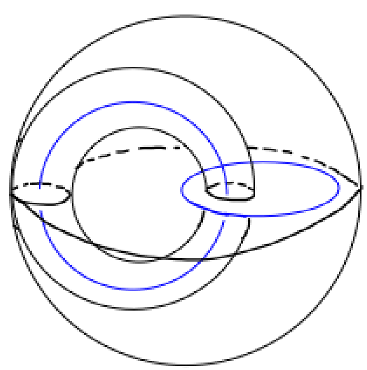


Figure 2.23: The deformation retractions of $\pi_1(\mathbb{R}^3\backslash(A\cup B))$

Definition 19. *A torus knot $K_{m,n} \subset S^3$ with $hcf(m,n) = 1$, is a knot pointed on $S^1 \times S^1$ wrapping n times around the first S^1 and m times around the second S^1.*
We now consider the fundamental group associated with a torus knot.

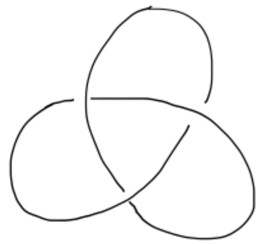


Figure 2.24: A trefoil knot

Example 23.

$$\pi_1(K_{m,n}) = \langle a, b | a^m = a^n \rangle$$

Proof. Consider the deformation retraction of $S^3 \backslash K_{m,n}$ on the space $X_{m,n} = S^1 \times [0,1] \cup_f S^1 \times \{0,1\}$
where $f : A = S^1 \times \{0,1\} \to S^1 \times \{0,1\}$

$$\pi_1(S^3 \backslash K_{m,n}) = \langle a, b | a^m = b^n \rangle = G$$

We can argue that G is a Ccentral extension $0 \to \mathbb{Z} \to G \to {}^{\mathbb{Z}}/_{m\mathbb{Z}} * {}^{\mathbb{Z}}/_{n\mathbb{Z}} \to 0$ G determines m, n (See Hatcher for more details).

□

Example 24. Surfaces of genus g

$$\pi_1 = (\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g | [a_1, b_1], [a_2, b_2], \ldots, [a_g, b_g] \rangle$$

given G one can always abelianise it i.e take $G^{ab} = {}^{G}/_{\text{commutators}}$ which leaves

$$\pi_1(\Sigma_g)^{ab} = \mathbb{Z}^{2g}$$

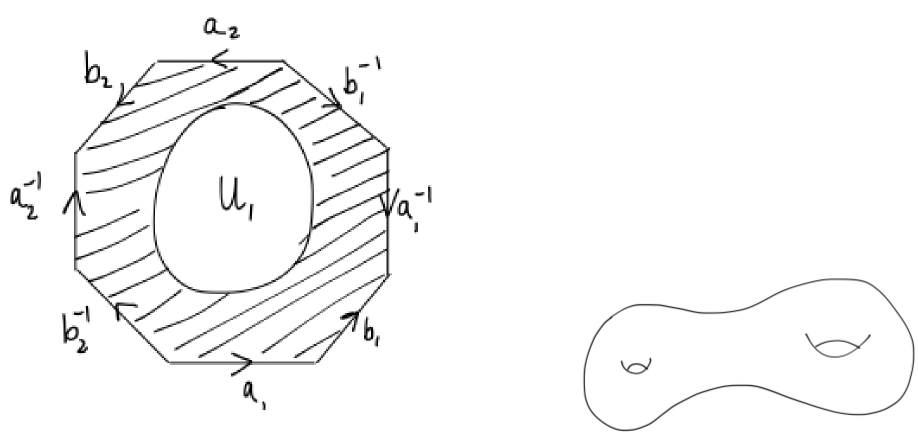


Figure 2.25: A genus 2 surface

Chapter 3

Covering Space Theory

Throughout the whole of this section we will assume that $p : \tilde{X} \to X$ is a covering map, where $\tilde{X}$ and X are both path connected and locally path connected. We will also assume that X is semi-locally simply connected (see definition below). The aim of this section is, for a fixed X, is to understand all of the covering spaces $\tilde{X}$ such that there exists $p : \tilde{X} \to X$. The following definitions are needed in the statement of the main theorem of this section- the fundamental theorem of covering spaces.

Definition 20. *X is semi-locally simply connected if $\forall x \in X$ there exists a neighbourhood of x such that;*

$$x \in U \overset{j}{\hookrightarrow} X$$

such that;

$$j_* : \pi(U, x) \to \pi_1(X, x)$$

is the trivial homomorphism.

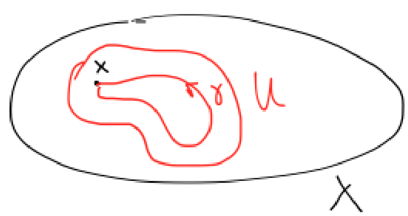


Figure 3.1: A semi-locally connected set

We now introduce the concept of automorphisms of covers.

Definition 21. *A **deck transformation** of $p : \tilde{X} \to X$ is a commutative diagram such that $p \circ \phi = p$ where $\phi : \tilde{X} \to \tilde{X}$ is a homeomorphism.*

Denote;

$$G(\tilde{X}) = G(\tilde{X} \big/ X)$$

be the group of all deck transformations.

The following definitions are essential to stating the fundamental theorem.

$$
\begin{tikzcd}
\tilde{X} \arrow[rr, "\phi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\
& X &
\end{tikzcd}
$$

Definition 22. *Suppose that $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, $q : (\tilde{Y}, \tilde{y}_0) \to (X, x_0)$ are coverings, then a **pointed isomorphism** is a pointed homeomorphism, $\phi : (\tilde{X}, \tilde{x}_0) \to (\tilde{Y}, \tilde{y}_0)$ such that $q \circ \phi = p$.*

*Similarly, suppose $p : \tilde{X} \to X$, $q : \tilde{Y} \to Y$ are coverings, then an **isomorphism** is a homeomorphism $\phi : \tilde{X} \to \tilde{Y}$ s.t. $q \circ \phi = p$.*

Definition 23. *A covering $p : \tilde{X} \to X$ is **normal** if for all $x \in X$ and for all $\tilde{x}_!, \tilde{x}_2 \in \tilde{X}$ s.t. $p(\tilde{x}_1) = p(\tilde{x}_2) = x$, there exists a $g \in G(\tilde{X}/X)$ such that $g(x_1) = x_2$.*

i.e. a covering is normal if for every point in X there is a deck transformation mapping any two of it's preimages under p.

3.1 The Fundamental Theorem of Covering Spaces

We are now ready to state the Fundamental Theorem of Covering Spaces.

Theorem 15. *Assume that X satisfies the assumptions outlined at the start of this section.*
Part (I)
There exists a bijection such that;

$$\left\{\text{coverings } p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)\right\} \Big/ \text{p'ted isomorphism} \leftrightarrow \left\{\text{set of subgroups} H \subset \pi_1(X, x_0)\right\}$$

(Where $p : (\tilde{X}, \tilde{x}_0) \to (X, x) \mapsto H = p_* \pi(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x)$)
However it is possible to ignore the basepoints to give;

$$\left\{\text{coverings } p : \tilde{X} \to X\right\} \Big/ \text{isomorphism} \leftrightarrow \left\{\text{conjugacy classes of } H \subset \pi_1(X, x_0)\right\}$$

Part (II)
*(A) Also $p : \tilde{X} \to X$ is **normal** $\Longleftrightarrow H \subset \pi_1(X, x_0)$ is a normal subgroup.*

(B) In general;

$$G(\tilde{X}/X) = N(H)/H$$

*Where $N(H) = \{g \in H \big| gHg^{-1} = H\}$ that is the **normalise** of H.*

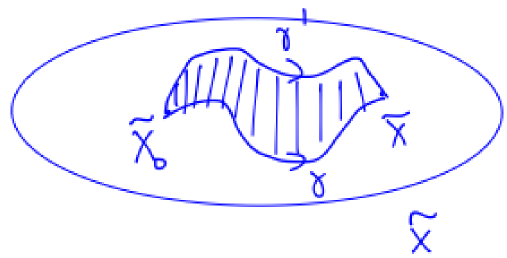
So if H is normal then;

$$G(\tilde{X}/X) = \pi(X, x_0)/H$$

Proof. Fix (X, x_0) path connected, locally path connected, semilocally simply connected. Given H we need somehow to 'cook up' $(\tilde{X}, \tilde{x_0}) \xrightarrow{p} (X, x_0)$. The most important case is $H = (e)$, which we will now consider.

We want to construct a simply connected covering (i.e with $\pi_1(\tilde{X}, \tilde{x_0}) = \{e\}$). This cover is called the **universal cover**.

What are the 'points' of $\tilde{X}$?



Consider the following correspondence, which is given by $[\gamma] \mapsto \gamma(1)$.

$$\left\{\text{points } \tilde{x} \in \tilde{X}\right\} \leftrightarrow \left\{\text{homotopy classes } [\gamma] \text{ of paths } \gamma : I \to \tilde{X} \text{with } \gamma(0) = \tilde{x_0}\right\}$$

Which is equivalent to,

$$\left\{\text{points } \tilde{x} \in \tilde{X}\right\} \leftrightarrow \left\{\text{homotopy classes } [\alpha] \text{ of paths } \alpha : I \to X \text{with } \gamma(0) = x_0\right\}$$

where the correspondence is given by $[\gamma] \subset \tilde{X} \mapsto [p \circ \gamma] \subset X$ on X.
Moreover the following correspondence, given by $\gamma \mapsto p \circ \gamma$, also holds.

$$\left\{\text{Homotopy classes } [\gamma] \text{ of paths in } \tilde{X} \text{with } \gamma(0) = \tilde{x_0}\right\} \leftrightarrow \left\{\text{homotopy classes } [\alpha] \text{ of paths } \alpha : I \to X \text{with } \gamma(0) = x_0\right\}$$

This is a bijective correspondence: it is surjective as paths can be lifted and injective as homotopies can be lifted.

We define $\tilde{X} = \{[\gamma] | \gamma : I \to X \text{ with } \gamma(0) = x_0\}$ and $p : \tilde{X} \to X$ is defined by $[\gamma] \mapsto \gamma(1) \in X$

What do I need to do?

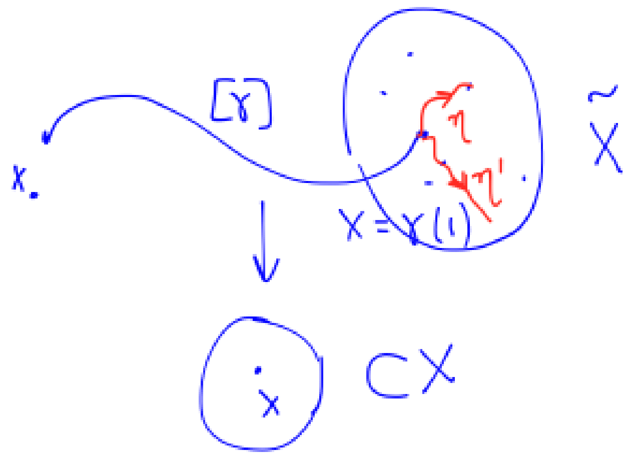
1. Endow $\tilde{X}$ with a topology and verify that p is a covering map
2. Show $\pi_1(\tilde{X}, \tilde{x_0}) = (e)$

In order to prove 1 we consider:

$$\mathcal{U} = \{U \subset X \text{ open } | U \text{ is path connected and } J_* : \pi_1(U, u) \to \pi_1(X, u)\ j_* = 0\}$$

Claim 1: by the assumption of semi-locally simply connected $\mathcal{U}$ is a basis for the topology of X. We don't prove this completely, let just verify 2.

Pick any $z \in U_1 \cap U_2$ and let $z \in V \subset U_1 \cap U_2$ be path connected and $j : v \hookrightarrow x \implies j_* = 0$.
Therefore $V \in \mathcal{U}$
How to put a topology on $\tilde{X}$?



Let $[\gamma] \in \tilde{X}$, $x = p([\gamma]) = \gamma(1)$ choose $x \in U \overset{i}{\hookrightarrow} X$ with $j_* = 0$

$$U_{[\gamma]}\{[\gamma \cdot \eta] | \eta : I \to U\}$$

Claim 2: the set of $U_{[\gamma]}$ is the basis for a topology on $\tilde{X}$

Note: $U_{[\gamma]} = U_{[\gamma']} if [\gamma'] \in U_{[\gamma]}$ indeed $[\gamma'] = [\gamma\eta]$ need a bijection $U_{[\gamma]} = U_{[\gamma']}$

$$[\gamma(\eta\eta')] = \gamma'\eta'$$

$$[\gamma\eta'] = [(\gamma\eta)(\bar{\eta}\eta')]$$

Claim 2 is not proved completely. We check 2 (for a basis) holds given $U_{[\gamma]}, V_{[\gamma']}$ & $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$
This means $[\gamma''] = [\gamma\eta] = [\gamma'\eta']$ in particular $U_{[\gamma]} = V_{[\gamma']}$ & $V_{[\gamma]} = V_{[\gamma']}$.

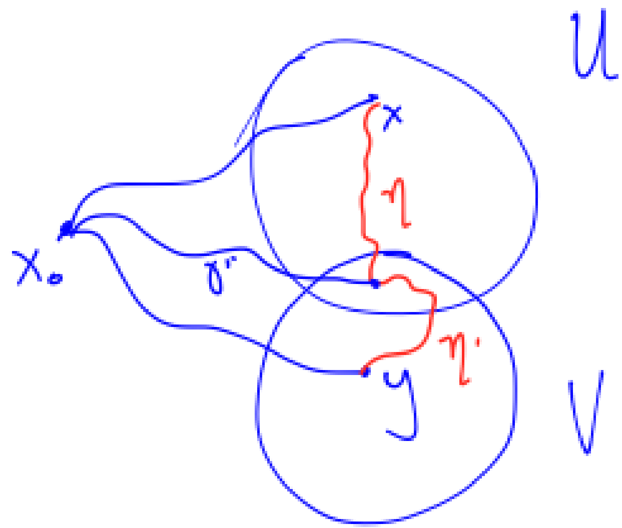
Take $z = \gamma''(1) \subset W \subset U_1 \cap U_2$ with $j_* = 0$, $j : W \hookrightarrow X$ then $W_{[\gamma'']} \subset U_{[\gamma'']} = U_{[\gamma]} \subset V_{[\gamma'']} = V_{\gamma'}$.
This puts a topology on $\tilde{X}$. p is obviously a covering map.

$$p : U_{[\gamma]} \xrightarrow{\simeq} U$$

This concludes the proof of (1).
It remains to show that $(\tilde{X}, \tilde{x_0})$ is path connected & $\pi_1(\tilde{X}, \tilde{x_0}) = \{e\}$
Fix path $\gamma : I \to X$, $\gamma(0) = x_0$. Consider path in $\tilde{X}$ $\Gamma : [0,1] \to \tilde{X}$.

$$\Gamma : [0,1] \ni s \mapsto [\gamma_s] \text{ where } \gamma_s(t) = \begin{cases} \gamma(t) & t \leq s \\ \gamma(s) & t \geq s \end{cases}$$

Then $\Gamma(0) = [x_0]$, $\Gamma(1) = [\gamma]$, $\Gamma(s) = [\gamma_s]$ & its endpoints $\gamma(s)$
So $p(\Gamma(s)) = \gamma(s)$ i.e $\Gamma : I \to \tilde{X}$ is the lift of $\gamma : I \to X$. Observe that Γ goes from $[x_0]$ to an arbitrary $[\gamma]$ therefore $\tilde{X}$ is path connected.



To see $\pi_1(\tilde{X}, \tilde{x_0} = \{e\}$ (because in general p_* is injective).

Suppose γ is a loop at x_0 in $p_*\pi_1(\tilde{X}, \tilde{x_0})$ this means that γ lifts to a loop in $\tilde{X}$ at $\tilde{x_0}$. This means that Γ is a loop in $\tilde{X}$. So $\Gamma(1) = [\gamma] = \Gamma(0) = [x_0]$.

We now consider the case for a more general $H \subset \pi_1(X, x_0)$. To do this we need to construct a cover q such that;

$$(X_H, x_H) \xrightarrow{q} (X, x_0)$$

We now sketch a proof in this case. Firstly, let $(\tilde{X}, \tilde{x}_0)$ be the universal cover just constructed. And define an equivalence relation R on $\tilde{X}$ by;

$$[\gamma] \sim_R [\gamma'] \iff \gamma(1) = \gamma'(1) \text{ and the loop } [\gamma\bar{\gamma}'$$

(Recall that $\bar{\gamma}$ is the 'inverse' loop.)
Then if $[\gamma] \sim_R [\gamma']$, $X_H = \tilde{X}\big/R$ has the quotient topology. And so;

$$\left(U_{[\gamma]} \times U_{[\gamma']}\right) \cap R$$

is the graph of the homeomorphism and $U[\gamma] \simeq U_[\gamma']$.
Therefore $p : X_H \to X$ is a covering map.

WE now check that,

□

Remark 1. *One can go further and define;*
(1) a category of coverings of X where the morphisms are diagrams s.t. Where ϕ is continuous and $q \circ \phi = p$ (2) In algebra, it is possible to make a category out of the "lattice" of subgroups of a group G. If $H_1, H_2 \subset G$ a subgroup, then a morphism $\phi : H_1 \to H_2$ is...
(3)Express the fundamental theorem as an equivalence of these 2 categories.

Lemma 1. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ covering, then $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$*

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\phi"] \arrow[dr, "q"'] & & \tilde{Y} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Proof. Suppose $\gamma : I \to \tilde{X}$ is a loop at $\tilde{X}_0$, and suppose that $p \circ \gamma : I \to X$ is homotopic to e_{X_0}. Then there exists a homotopy, $F : I \times I \to X$ such that;

$$\begin{cases} F(0,t) & = p \circ \gamma(t) \\ F(1,t) & = x_0 \\ F(s,0) & = x_0 \\ F(s,1) & = x_0 \end{cases}$$

This can be seen graphically in figure 3.2. And so by the homotopy lifting theorem, the homotopy

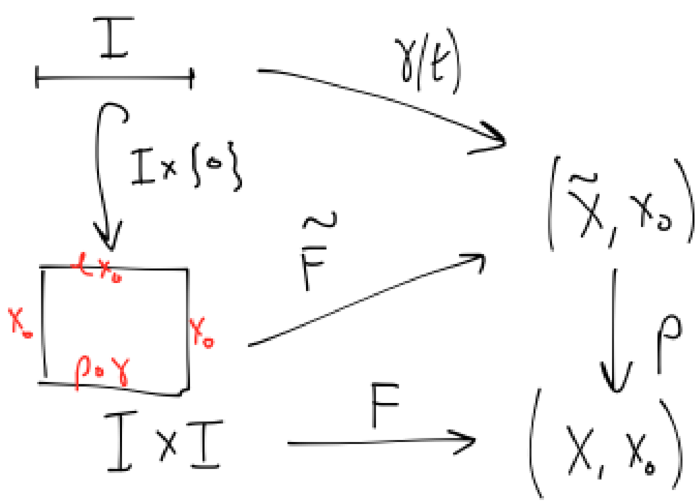
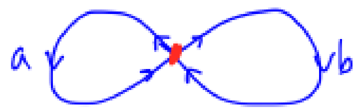


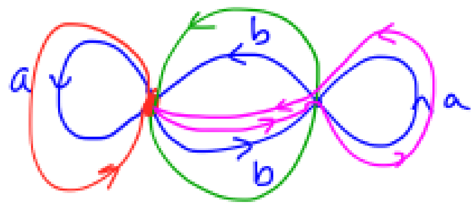
Figure 3.2: Graphical representation of the proof of lemma 1

from $p \circ \gamma$ to e_{X_0}, lifts to a unique homotopy form γ to $e_{\tilde{X}_0}$ in $\tilde{X}$ $\square$

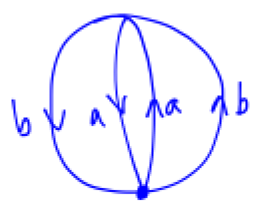
Example 25. *Coverings $S^1 \vee S^1$. Recall that $S^! \vee S^1$ has fundamental group $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$ and can be seen graphically as;*



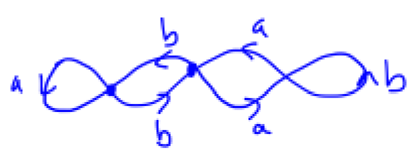
1. *Where the subgroup $H = \langle a, b^2, bab^{-1} \rangle \subset \mathbb{Z} * \mathbb{Z}$.*



2. *In this case the subgroup $H = \langle a^2, b^2, ab \rangle$.*



3. *Therefore $H = \langle a^2, b^2, aba, bab \rangle$.*



Chapter 4

Homology

4.1 Simplicial Homology

We now introduce the concept of Homology. See Hatcher for an intuitive introduction to this theory.
Definition 24. *The standard n-simplex is;*

$$\Delta_n = \{(t_0, \ldots, t_m) \Big| t_i \geq 0, \sum_i t_i = 1\} \subset \mathbb{R}^{n+1}$$

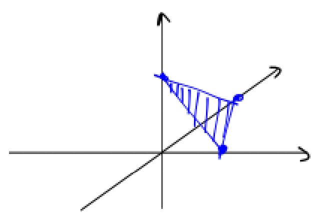


Figure 4.1: The standard simplex $\Delta_2 \subset \mathbb{R}^3$.

Example 26.
Definition 25. *A n-simplex $\subset \mathbb{R}^m$ is $conv\{v_0, \ldots, v_m\}$, 'the convex hull', where v_i are not all contained in an affine subspace of dimension $< n$. The v_i are known as vertices.*

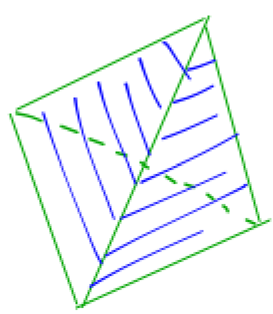


Figure 4.2: A 3-simplex in $\mathbb{R}^3$

We will work with **ordered** n-simplices, that is a n-simplex with a specified ordering of the vertices. We will denote this by, $[v_1, \ldots v_n]$.

There exists a natural affine isomorphism $\Delta_n \to [v_0, .., v_n]$ where $(t_0, \dots, t_n) \to \sum_i t_i v_i$.

Remark 2. *The standard simplex is ordered, $[e_0, \dots, e_n] \subset \mathbb{R}^{n+1}$ where e_i is the ith standard basis vector.*

Definition 26. *The **facets** of $[v_0, .., v_n]$ are the ordered $(n-1)$-simplices;*

$$\Delta^i_{n-1} = [v_0, \dots, \hat{v}_i, \dots, v_n$$

Where $\hat{v}_i$ means omit v_i.

Definition 27. *A Δ-complex structure on a space X is a collection of continuous maps;*

$$\sigma_\alpha : \Delta_n \to X$$

Such that the following hold.

1. *$\sigma_\alpha|\mathring{\Delta}_n : \Delta_n \to X$ is injective and for all $x \in X$ there exists a unique $\alpha : x \in \sigma_\alpha(\Delta_n)$*
2. *If $\Delta_{n-1} \subset \Delta_n$ is a facet then $\exists \sigma_\alpha|\Delta_{n-1} = \sigma_\beta$*
3. *$U \subset X$ is open if and only if for all $\sigma_\alpha^{-1}(U) \subset \Delta_n$ are open*

Note we define the **topological boundary** to be;

$$\partial\Delta_n = \cup_{i=0}^n \Delta^i_{n-1}$$

And the **topological interior** to be;

$$\mathring{\Delta}_n = \Delta \backslash \partial\Delta_n = \{\sum t_i r_i \Big| 0 < t_i \text{ for } i = 0, \dots, n \text{ and } \sum t_i = 1\}$$

We now give examples of a few constructions where we impose a Δ-simplex structure, in order to calculate the homology groups.

Example 27. 1. *T = torus;*

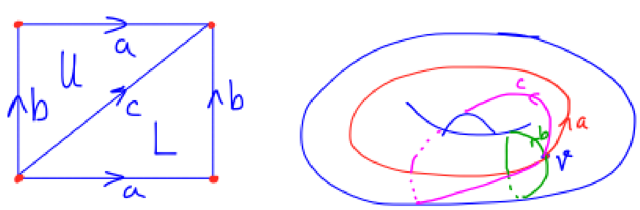


Figure 4.3: the Δ-simplex structure on a Torus

This is now a Δ-complex structure where there are; 1 0-dimensional simplex $[v]$, 3 1-dimensional simplices a, b, c and 2 2-dimensional simplices U, L.

2. $\mathbb{P}^2(\mathbb{R})$

   This is also a Δ complex structure with 2 0-dimensional simplices, 3 1-dimensional simplices and 2 2-dimensional simplices.

3. *Similarly, consider the Klein Bottle.*

   Where there is 1 0-dimensional simplex, 3 1-dimensional simplices and 2 2-dimensional simplices.

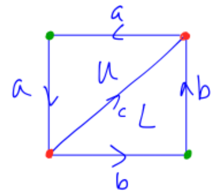


Figure 4.4: Δ-simplex structure on $\mathbb{P}^2(\mathbb{R})$

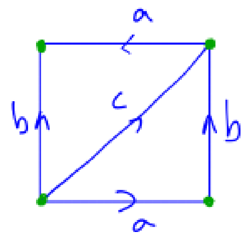


Figure 4.5: The Δ-simplex structure on the Klein bottle

4. *A similar contruction can be applied to a surface of genus g.*

We note that;

- X has a quotient topology $X = {\sqcup \Delta_n^\alpha}/{\sim}$
- X is always Hausdorff
- A Δ-complex is a very special kind of CW complex.

We now construct the homology groups $H_n(X)$.

Definition 28. *A **complex of abelian groups** is a diagram;*

$$\cdots \to C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} \ldots$$

where, C_i is an abelian group for each $i \in \mathbb{N}$.
Each $\partial_i : C_i \to C_{i-1}$ is a homeomorphism for all $i \geq 1$, and $\partial_n \circ \partial_{n-1} = 0$ for all n.

Using this definition it is possible to define the Homology groups as follows.

Definition 29. *Let the n-cycles be denoted by*

$$Z_n = ker(\partial_n) \subset C_n$$

and the n- boundaries be,

$$B_n = Im(\partial_{n+1}) \subset C_n$$

Then the n-homology, H_n is;

$$H_n = {Z_n}/{B_n}$$

It is possible to define a complex of abelian groups out of a Δ-complex. This is done as follows, let C_n^Δ be the free abelian group with basis the n-simplices, Δ_n^α, of the complex Δ. An element of C_n^Δ can be expressed as $\sum n_\alpha \Delta_n^\alpha$, with $n_\alpha \in \mathbb{Z}$. (We require that all but finitely many of these n_α are equal to 0.)

Then define the homeomorphisms $\partial_i : C_i^{\Delta} \to C_{i-1}^{\Delta}$ to act on the basis as;

$$\partial(\Delta_n^{\alpha}) = \sum_{i=1}^{n}(-1)^i \Delta_n^{\alpha,i}$$

We then have the following lemma.
Lemma 2. *$\partial_i^2 = 0$ for all i.*

Proof. We prove this by direct calculation.

$$\begin{aligned}\partial\partial[v_0,\ldots,v_n] &= \partial\sum_i(-1)^i[v0,\ldots,\hat{v}_i,\ldots v_n] \\ &= \sum_{j<i}(-1)^i[v_0,\ldots,\hat{v}_j,\hat{v}_i,\ldots,v_n] + \sum_{j>i} \\ &= 0\end{aligned}$$

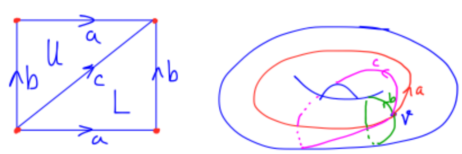
□

We can then define that;

$$H_n^{\Delta}X := H_n(C_n^{\Delta}(X))$$

We now use this method to compute the Homology groups for the torus, the projective plane, the Klein bottle and for a surface of genus g.

Example 28. *Apply the Δ-complex structure onto the torus as done previously.*



This gives, 2 2-cells, 3 1-cells and 1 1-cell, this structure produces the sequence of;

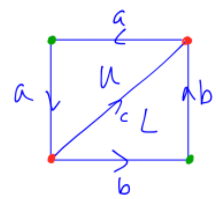
$$0 \xrightarrow{\partial_3} C_2^{\Delta}(X) \xrightarrow{\partial_2} C_1^{\Delta}(X) \xrightarrow{\partial_1} C_0^{\Delta}(X) \xrightarrow{\partial_0} 0$$

Consider the map ∂_1. Then for all 1-cells, a, b, c, we get $\partial_1(a) = v - v = 0$ similarly $\partial_1(b) = \partial_1(c) = 0$, and so $\partial_1 = 0$. And hence $C_0^{\Delta} \simeq \mathbb{Z}$.

Now consider the map ∂_2. $\partial_2(U) = c - a - b$ and $\partial_2(L) = a + b - c$. Hence $\{c, a + b - c\}$ forms a basis for Δ_1 and so it follows that $H_1^{\Delta}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Moreover since the Δ-complex structure contains no 3-simplices the higher simplicial homology groups are zero. Moreover $H_2^{\Delta}(T) \simeq ker(\partial_2)$, which is an infinite cycle generated by $U - l$. Since $\partial_2(pU + qL) = (p+q)(a+b-c) = 0$ if and only if $p = -q$. Therefore the simplicial homology groups of the torus are;

$$H_n^{\Delta}(X) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for n ¿ 2} \end{cases}$$



Example 29. *We will now consider the homology groups for the projective plane. Consider the Δ-complex structure that can be applied onto the projective plane, that is consider the projective plane to be; Then this induces the following sequence of groups.*

$$0 \xrightarrow{\partial_3} C_2^{\Delta}(X) \xrightarrow{\partial_2} C_1^{\Delta}(X) \xrightarrow{\partial_0} 0$$

Let the 1 cells be vertices v_0, v_2. Then;

$$H_0^{\Delta} = {}^{Ker\partial_0}\big/_{Im\partial_1} = {}^{\langle v_0, v_1\rangle}\Big/_{v_1 - v_0}$$

since $\partial_1(a) = v_1 - v_0, \partial_1(b) = v_1 - v_0$. Therefore, $H_0^{\Delta}X = \mathbb{Z}$.

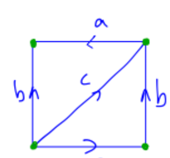
$$H_1^{\Delta}X = \frac{\langle a+b, c\rangle}{\langle a+b+c, a+b-c\rangle} = \frac{\langle a+b\rangle}{2\langle a+b\rangle} \simeq {}^{\mathbb{Z}}\big/_{2\mathbb{Z}}$$

And;

$$H_2 = ker\partial_2 = (0)$$

And all other groups are (0).

Example 30. *Consider now the homology groups of the Klein bottles. Then this structure gives*



the following sequence;

$$0 \xrightarrow{\partial_3} C_2^{\Delta}(X) \xrightarrow{\partial_2} C_1^{\Delta}(X) \xrightarrow{\partial_0} 0$$

Where ∂_2 is given by the matrix;

$$\partial_2 \begin{pmatrix} U \\ L \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} U \\ L \end{pmatrix}$$

Where U and L are the 2-cells in complex. And $\partial_1 = 0$, by arguments similarly to above.

$$H_0^{\Delta}K \simeq \mathbb{Z},\ H_1^{\Delta}K \simeq {}^{ker\partial_1}\big/_{im\partial_2} = \frac{\langle a, b, c\rangle}{a = b + c} = \frac{\langle a, b\rangle}{\langle 2b\rangle} \simeq \mathbb{Z} \oplus {}^{\mathbb{Z}}\big/_{2\mathbb{Z}}$$

4.2 Singular Homology

We now aim to define the Homology groups for more general spaces than the Δ-complexes.

Definition 30. *Let X be a complex space, a **singular n-simplex** in X is a continuous map $\sigma : \Delta_n \to X$.*
We form a $C.X$:

$$C_n X = \bigoplus_{\sigma:\Delta_n \to X} \mathbb{Z}[\sigma]$$

(a n-chain) singular simplex. An element is a formal linear combination:

$$c = \sum_{\sigma:\Delta_n \to X} u_\sigma \sigma$$

where $u_\sigma \in \mathbb{Z}$, finitely many of them are 0.

Let $\partial_n : C_n X \to C_{n-1} X$ where:

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i [\sigma|[v_0 \dots \hat{v_i} \dots v_n]]$$

(Then it's clear that $\partial^2 = 0$.)

$$H_n^* X = {}^{Z_n}\big/_{B_n}$$

Recalling that; Z_n = cycles = $Ker(\partial_n)$, B_n = boundaries = $Im(\partial_{n+1})$.

Moreover notice that; $B_n \subset Z_n$ because $\partial^2 = 0$.

We now introduce a few fundamental properties.

Proposition 5.

1. *If $X = \coprod X_\alpha$ with X_α path-connected then $H_n X = \bigoplus H_n X_\alpha$*
2. *If $X \neq \emptyset$ and path-connected then $H_0 X = \mathbb{Z}$*
3. *If $X = \{pt\}$ then $H_n X = \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ (0) \text{ otherwise} \end{cases}$*
4. *Basic functoriality: a continuous map $g : XtoY$ induces $f_* : H_n X \to H_n Y \; \forall n$ which statisfies the functorial property i.e $(f \circ g)_* = f_* \circ g_*$*

Proof.

1. This is obvious since
$$C_n = \bigoplus_\alpha C_n X_\alpha$$
(A simplex is path connected. A singular simplex $\sigma : \Delta \to X$ must land in one and only one of the path components X_α).

2. $H_* X = {}^{C_0(X)}\big/_{Im(\partial_1)}$ Note that $C_0 X = \bigoplus_{x \in X} \mathbb{Z}.x$

   Define $\epsilon : C_0 X \to \mathbb{Z}$ by,
$$\epsilon \left(\sum_{i=1}^{k} n_i x_i \right) = \sum_{i=1}^{k} n_i \in \mathbb{Z}$$

Then it is clear that $Im(\partial_1) \subset Ker(\epsilon)$. And so we claim that if $Im(\partial_1) = Ker(\epsilon)$ then $\epsilon : H_0(X) \simeq \mathbb{Z}$.

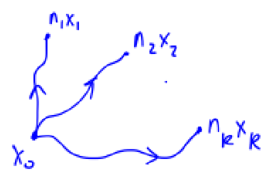
Consider

$$z = \sum_{i=1}^{k} n_i x_i \in Ker(\epsilon)$$

That is $\sum_{i=1}^{k} n_i = 0$, then choose $x_0 \in X$. And so;

$$\sigma_i : I \to X, \sigma_i(0) = x_i, \sigma_i(x) = x_i$$

Then these σ_i are singular 1-simplexes.



$$\begin{aligned}
\partial\left(\sum_{i=1}^{k} n_i\sigma_i\right) &= \sum_{i=1}^{k} n_i\partial(\sigma_i) \\
&= \sum_{i=1}^{k} n_i(x_i - x_0) \\
&= \sum_{i=1}^{k} un_i x_i - \sum_{i=1}^{k} un_i x_0 \\
&= \sum_{i=1}^{k} x_i = z
\end{aligned}$$

Therefore $z \in Im(\partial_1)$.

3. If $X = \{pt\}$ $C_n X = \mathbb{Z}$ and $\partial_n = \begin{cases} 0 & \text{if } n = \text{odd} \\ 1 & \text{if } n = \text{even} \end{cases}$

Then this implies that;

$$H_n\{pt\} = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ (0) & \text{otherwise} \end{cases}$$

(4) If $\sigma : \Delta_n \to X$ is a singular simplex in X then;

$$f_*\sigma := f \circ \sigma : \Delta_n \to Y$$

is a singular simplex in Y. This produces the following commutative diagram;

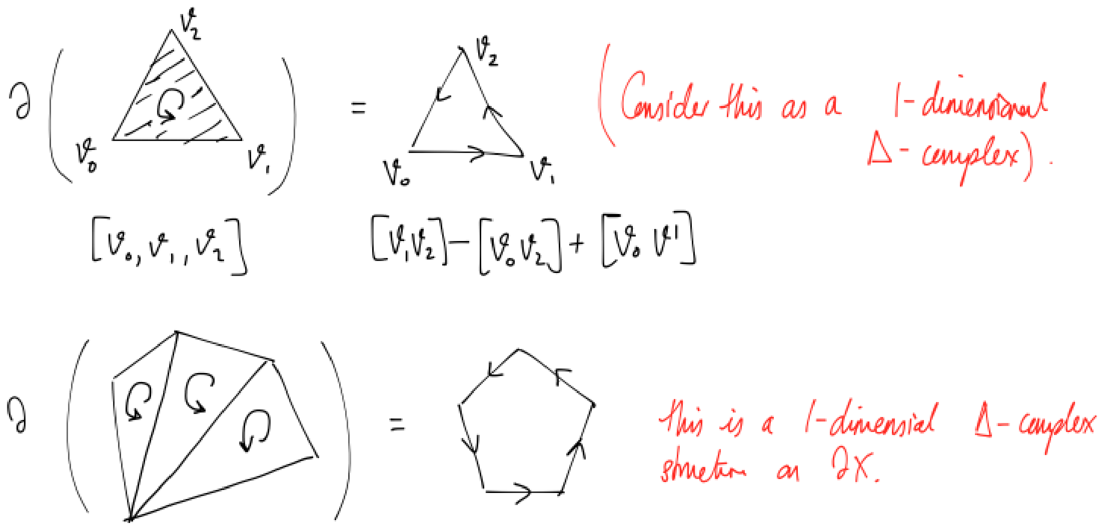
$$\begin{tikzcd}
{} \arrow[r] & C_2(X) \arrow[r] \arrow[d, "f_*"] & C_1(X) \arrow[r] \arrow[d, "f_*"] & C_0(X) \arrow[d, "f_*"] \\
{} \arrow[r] & C_2(Y) \arrow[r] & C_1(Y) \arrow[r] & C_0(Y)
\end{tikzcd}$$

That is $f_* \circ \partial = \partial \circ f_*$. And so, $f_*(B_n X) \subset B_n Y$, $f_*(Z_n X) \subset Z_n Y$, and this implies that;

$$f_* : H_n X \to H_n Y$$

□

An informal discussion on visualising cycles and boundaries.



An orientated 2-dimensional Δ-complex with boundary structure on $(Y, \partial Y)$ and a continuous map, $f : Y \to X$, 'gives' a chain, $[f] \in C_2 X$ which can be expressed as;

$$[f] = \sum_{\sigma \in \Delta^{[2]}} f|_\sigma, \partial[f] = [f|_{\partial Y} \in C_1 X, \partial[f] = 0 \text{ if } \partial Y = \emptyset$$

Consider a 2-chain in X, then a 2-chain is a boundary if $\partial Y = \emptyset$.

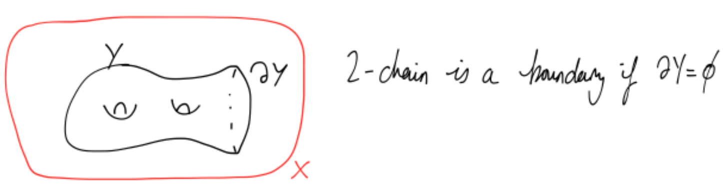


Figure 4.6: 2-chain is a boundary if $\partial Y = \emptyset$

Consider the image of a trivial 1-homology class on X. Then the 1-cycle = ∂(2 chain with boundary).

Remark 3. It's easy to see that a 2-chain/2-cycle etc. . . is a formal sum of things of this sort. However, this works for $C_n X, n \geq 3$ in these pictures Y was a manifold but for $n \geq 3$ we need to allow Y to be 'singular' in codimension ≥ 3.

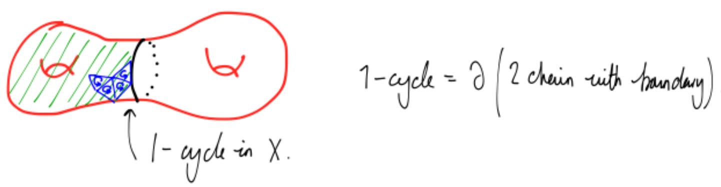


Figure 4.7: 1-cycle = ∂(2 chain with boundary)

4.2.1 Homotopy Invariance of Singular Homology

We aim to prove that homotopic spaces have isomorphic Homology Groups, this will be done by showing that a map $f : X \to Y$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for each n and that f_* is an isomorphism if f is a homotopy equivalence. But we begin with the following definition;

Definition 31. *A **chain map** between complexes K, L is a group homomorphism $f : K \to L$ such that $f \circ \partial = \partial \circ f$.*

We remark that in the case when $K = C_n(X)$ and $L = C_n(y)$ for some X, Y say then the property that $\partial f = f\partial$ implies that f sends cycles to cycles and boundaries to boundaries. Therefore this has the property that $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$. This can be stated more formally as;

Lemma 3. *A chain map induces a homomorphism $f_* : H_nX \to H_nY$.*

For clarity it is useful to define what homotopic means in the context of chain maps.

Definition 32. *Then $f_., g_. : K \to L$ are **homotopic** if there exists a prism operator $p_. : K_. \to L_{.+1}$ such that $\partial P + P\partial = g - f$.*

Then we can formulate the main theorem of this section that,

Theorem 16. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*
Moreover these induced maps f_* are isomorphisms.

Proof. □

An application of this theorem is the following.
Example 31. Suppose X is contractible then $\tilde{H}_n(X) = 0$ for all n. This is since X is homotopic to a point set and so these spaces must have isomorphic homology groups.

4.3 Reduced Homology

Definition 33. *A pair X, A is **good** if there exists an open set $A \subset U \subset X$ such that A is a deformation retract of U.*

We define the **reduced homology** of X to be the homology of the complex of augmented singular classes of X, where;

$$\widetilde{C}_n X = \begin{cases} C_n X & \text{if } n \neq -1 \\ \mathbb{Z} & \text{if } n = -1 \end{cases}$$

And from this it implies that;

$$\widetilde{H}_n X = \begin{cases} H_n X & \text{if } n \neq -1 \\ ker(H_0 X \xrightarrow{\epsilon} \mathbb{Z}) & \text{if } n = -1 \end{cases}$$

where $\epsilon : C_0 X \to C_{-1} X$ where $\epsilon(x) = 1$ for all x.

Example 32. *Suppose $x \in X$ is a point then $H_n(X, \{x\}) \simeq \widetilde{H_n X}$. This is more or less trivial since;*

$$H_0(\{x\}) = \begin{cases} (0) & \textit{if } n \neq 0 \\ \mathbb{Z} & \textit{if } n = 2 \end{cases}$$

And we have the following chain;

Which implies that;

$$H_0(X, \{x\}) \simeq \widetilde{H_0 X}$$

Theorem 17. *For X, A a good pair there exists a long exact sequence;*

$$\cdots \to \widetilde{H_n A} \to \widetilde{H_n X} \to \widetilde{H_n({X}/{A})} \to \widetilde{H_{n-1} A} \to \dots$$

Proof. This is more or less just the proof of the long exact sequence of the pair (X, A). The relative homology is easier to understand the quotient map;

$$q : (X, A) \to ({X}/{A}, {A}/{A} = \{pt\})$$

Which induces an isomorphism;

$$q_* : H_n(X, A) \to H_n({X}/{A}, \{pt\}) = \tilde{H}_n({X}/{A})$$

Let us prove that q_* is indeed an isomorphism.

Let $A \subset U \subset X$ be as in the definition of a good pair. Then by excision we have that;

$$H_n(X, U) \simeq H({X}/{A}, {U}/{A})$$

And;

$$H_n({X}/{A}) \simeq H_n(({X}/{A}) \backslash ({A}/{A}), ({U}/{A}) \backslash ({A}/{A}))$$

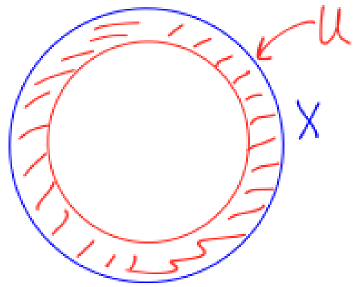
Since we have a good pair the following commutative diagram exists. Then by the long exact

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "i_*"] \arrow[d, "q_*"] \arrow[dr, phantom, "\circlearrowright"] & H_n(X,U) \arrow[r, "\simeq"] \arrow[d, "q'_*"] \arrow[dr, phantom, "\circlearrowright"] & H_n(X \backslash A, U \backslash A) \arrow[d, "q''_*"] \\
H_n({X}/{A}, {A}/{A}) \arrow[r, "j_*"] & H_n({X}/{A}, {U}/{A}) \arrow[r, "\simeq"] & H_n(({X}/{A}) \backslash ({A}/{A}), ({U}/{A}) \backslash ({A}/{A}))
\end{tikzcd}$$

sequence of the pair U, A it follows that i_* is an isomorphism. Similarly j_* is an isomorphism. Therefore q_* is also an isomorphism by the commutative property of the diagram above. □

Example 33. If $n \geq 1$ then;

$$H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$



To prove this we apply the previous result to $D^n, \partial D^n \simeq S^{n-1} = A$. Then;

$$X/A = D^n/\partial D^n = S^n$$

Then we have the following sequence;

$$\tilde{H}_n(D_n) \to \tilde{H}_n(S^n) \to \tilde{H}_{n-1}(D_n) \to \tilde{H}_{n-1}(D_n) \to \tilde{H}_{n-1}(S^n) \to \tilde{H}_{n-2}(S^{n-1}) \to \tilde{H}_{n-2}(D_n) \to \dots$$

Which becomes;

$$(0) \to \mathbb{Z} \to \mathbb{Z} \to (0) \to (0) \to (0) \to (0) \to \dots$$

4.4 Relative Homology

Let $A \subset X$ be a subspace then we can define,

Definition 34. ***Complex of relative n-chains*** *$C_{.}(X, A)$ be;*

$$C_n(X, A) = C_nX/C_nA$$

with the obvious induced boundary map of $\partial : C_{n+1}(X, A) \to C_n(X, A)$.

Definition 35. ***Homology of the pair*** *X, A;*

$$H_n(X, A) := H_nC_{.}(X, A), \frac{\{c \in C_nX | \partial C \in C_n(A)\}}{\{\partial C_{n+1}X + C_nA\}}$$

Figure 4.8 is a way of visualising the homology X, A.

There is an easy functoriality. Let $A \subset X$ be a subspace and $f : X \to Y$ is a continuous map such that $f(A) \subset B$, then;

$$f_* : H_n(X, A) \to H_n(X, B)$$

Then we get a **Homotopy Invariance**. Let $f : X \to Y$, $g : X \to Y$ as above and $f \sim g$ through the following map.

$$F : X \times I \to Y$$

and let $f_t : X \to Y$ be defined by $f_T(x) = F(x, t)$ where F is a homotopy through the maps of pairs and $f_t(A) \subset B$ for all t. This gives us $f_0 = f$, $f_1 = y$.
Then it follows that;

$$f_* = g_* : H_n(X, A) \to H_n(Y, B)$$

We then have two exact sequences, namely;

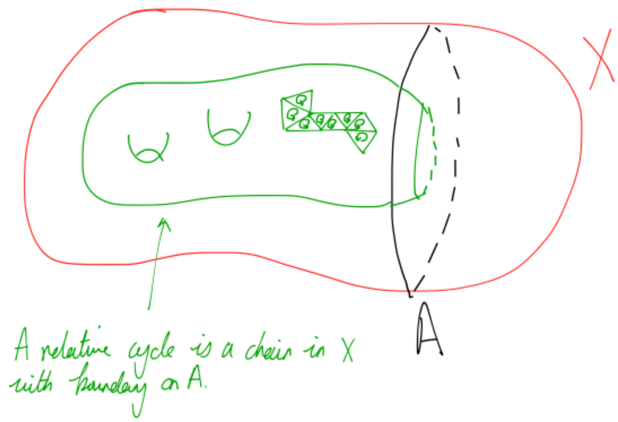


Figure 4.8: Relative Homology of X and A

- The **sequence of the pair**

$$\dots \to H_n A \to H_n X \to H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \to \dots$$

- The **sequence of the triple**

$$\dots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \xrightarrow{\delta} H_{n-1}(A, B) \to \dots$$

Exercise: fill out the details using the arguments similar to last time.

4.4.1 Barycentric Subdivision

We now provide the technical set-up required for a proof of the Excision Theorem which will be stated later.

Definition 36. *Let X be a topological space $\mathcal{U} = \{\mathcal{U}_i : i \in I\}$ is a collection of subspaces of X such that $\{\mathring{\mathcal{U}} : i \in I\}$ forms an open cover of X*
Set $C_n^{\mathcal{U}}(X) = \{$chains $\sum n_i \sigma_i$ in $C_n(X) | \sigma_i$ has image inside one of the $\mathcal{U}_i$ for each $i\}$ So

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X)$$

$$C_n^{\mathcal{U}}(X) \xrightarrow{\partial} C_{n-1}^{\mathcal{U}}(X)$$

Thus $(C_\cdot^{\mathcal{U}}(X), \partial)$ is a chain complex, in fact a subcomplex of $(C_\cdot(X), \partial)$

A philosophical point to be made here is that, when computing homology groups, we can insist that our simplices are small.

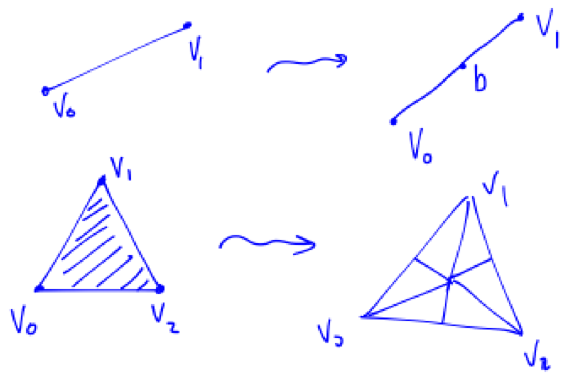
Proposition 6. *The inclusion $i : (C_n^{\mathcal{U}}, \partial) \to (C_n, \partial)$ is a chain homotopy equivalence. In other words $\exists$ a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}$ such that $\rho \circ i$ and $i \circ \rho$ are both chain homotopic to the identity. Thus i induces isomorphism:*

$$H_n^{\mathcal{U}}(X) \simeq H_n(X) \quad \forall n$$

We will prove the proposition above using barycentric subdivision.

Barycentric Subdivision of Simplices

Example 34. We give examples of the barymetric subdivision of standard 1 simplex and 2 simplex.



The aim of barymetric subdivision is to baricentrically subdivide the boundary and then add the barycentre.
The **barycentre** of $[v_0 \dots . v_n]$ is

$$\frac{1}{n+1}v_0 + \frac{1}{n+1}v_1 + \dots + \frac{1}{n+1}v_n$$

The **barycentric subdivision** of $[v_0 \dots . v_n]$ is the sum of simplices $[b, w_0 \dots . w_{n-1}]$ where b is the barycentre and $[w_0 \dots . w_{n-1}]$ is a simplex in the barycentric subdivision of $[v_0 \dots \hat{v}_i \dots v_n]$ for some i. In order to ensure that this inductive definition is well defined we require that the barycentric subdivision of $[v_0]$ is $[v_0]$.

The **vertices of barycentric subdivision** of $[v_0 \dots v_n]$ are; pick $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$, pick $k+1$ of the v_i's then

$$\frac{1}{k+1}v_{i_0} + \frac{1}{k+1}v_{i_1} + \dots + \frac{1}{k+1}v_{i_k}$$

is a vertex of barycentric subdivision.

Barycentric subdivision makes simplices smaller. Recall the following definition of diameter of a Δ-complex

$$\text{diam}(\Delta) = \sup d(x, y), \quad \Delta \subseteq \mathbb{R}^n, d(x, y) = |x - y|$$

Lemma 4.

$$diam(\Delta) = max|v_i - v_j|, \Delta = [v_0, \dots, v_n]$$

Proof. For $v \in \Delta$, $v = \sum_{i=0}^{n} t_i v_i$. So for all $w \in \Delta$.

$$\begin{aligned}
[h!]d(w,v) = |w - v| &= |w - \sum_{i=0}^{n} t_i v_i| \\
&= |\sum_{i=0}^{n} t_i (w - v_i)| \\
&= \sum_{i=0}^{n} t_i |w - v_i| \\
\leq \max_{j} |w - v_j|
\end{aligned}$$

Then apply this again decomposing w. □

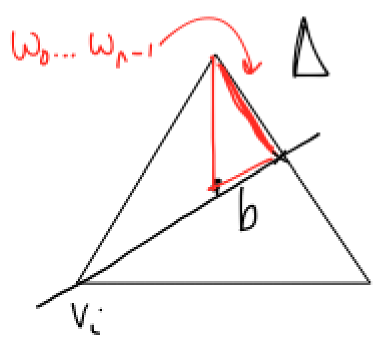
This lemma is useful to prove the following proposition.
Proposition 7. *Let Δ' be a simplex in the barycentric subdivision of Δ then;*

$$diam(\Delta') \leq \left(\frac{n}{n+1}\right) diam(\Delta)$$

Proof. We prove this by induction on n. The base case, when $n = 0$ holds and so. Assume that this is true for $n - 1$. Let $b =$barycenter of Δ. Then we have two cases:

- If b is not a vertex of Δ' then Δ' lies in $\partial(\Delta)$ and so we are done by induction.

- Suppose b is a vertex of Δ'. Then; Then $\Delta' = [bw_0 \dots w_{n-1}]$, where $[w_0 \dots w_{n-1}]$ is a sim-



plex in the barycentric subdivision of the face, $[v_0, \dots, \hat{v}_i, \dots, v_n]$ of Δ.

Then the line through v_i and b meets $[v_0 \dots \hat{v}_i \dots v_n]$ in the barycenter b_i of $[v_0 \dots \hat{v}_i \dots v_n]$.

Where $v_i = 0v_0 + \dots + 0v_{i-1} + v_i + 0v_{i+1} + \dots + 0v_n$ and $b = \frac{1}{n+1}v_0 + \dots + \frac{1}{n+1}v_{i-1} + \frac{1}{n+1}v_i + \dots + \frac{1}{n+1}v_n$. Then adding these and choosing k gives;

$$b = \frac{1}{n+1}v_i + \frac{n}{n+1}b_i$$

So,

$$|b - v_i| = \frac{n}{n+1}|v_i - b_i| \leq \frac{n}{n+1} diam(\Delta)$$

And so it follows;

$$diam(\Delta') = max|w_i - w_j| \leq \frac{n}{n+1} diam(\Delta)$$

where w_i, w_j are vertices of Δ.

$\square$

Barycentric Subdivision of Linear Chains

The goal now is to construct a function $p : C_n(X) \to C_n^u(X)$ that takes barycentric subdivision of chains.

Step 1: Suppose $Y =$ convex subset of $\mathbb{R}^n$. Then we can consider;

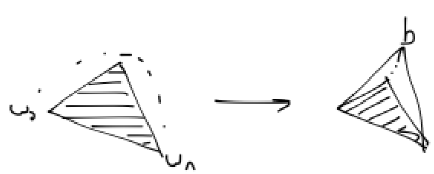
$$LC_n(Y) = \text{subcomplex in } C_n(Y) \text{ consisting of linear chains } \sigma : \Delta \to Y$$

We have that;

$$\begin{array}{ccc} C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) \\ \cup & & \cup \\ LC_n(Y) & \xrightarrow{\partial} & LC_{n-1}(Y) \end{array}$$

Since a linear map out of a subset is a linear map. So $(LC_{\cdot}(Y), \partial)$ is a subcomplex of $(C_{\cdot}(Y), \partial)$. Write $\sigma : [v_0, \dots, v_n] \to Y$ a linear chain as $[w_0 \dots w_n]$ where $w_i = \sigma(v_i)$. This determines σ uniquely.

Definition 37. *Given $b \in Y$, then there exists a cone operator $B : LC_n(Y) \to LC_{n+1}(Y)$ that maps $[w_0, \dots, w_n] \mapsto [bw_0, \dots, w_n]$*



Lemma 5.

$$\partial b([w_0, \dots, w_n]) = [w_0, \dots, w_n] - b(\partial[w_0, \dots, w_n])$$

Proof. This is proven by algebraic manipulation. $\square$

Proposition 8 (Key Properties).

- $S\partial = \partial S$
- $S \sim \mathbb{1}_{\widetilde{LC_n}} Y$

Proof. (a) Observe that, for $\lambda \in \widetilde{LC}_n Y$, $\lambda : \Delta_n \to Y$ is the generator.

$$\begin{aligned}
\partial S(\lambda) = \partial b_\lambda S(\partial\lambda) &= S(\partial\lambda) - b_\lambda S(\partial\lambda) \\
&= S\partial\lambda - b_\lambda \partial S(\partial\lambda) \\
&= S\partial\lambda - b_\lambda S(\partial \circ \partial\lambda) \\
&= S\partial\lambda
\end{aligned}$$

(b) We define $T : \widetilde{LC}_n Y \to \widetilde{LC}_{n+1} Y$ to be the homotopy from S to $\mathbb{1}$. Then we get; Then since

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{LC}_2 \arrow[r] \arrow[d] \arrow[dl] & \widetilde{LC}_1 \arrow[r] \arrow[d] \arrow[dl] & \widetilde{LC}_0 \arrow[r] \arrow[d] \arrow[dl] & \widetilde{LC}_{-1} \arrow[r] \arrow[d] \arrow[dl] & 0 \arrow[dl] \\
\cdots \arrow[r] & \widetilde{LC}_2 \arrow[r] & \widetilde{LC}_1 \arrow[r] & \widetilde{LC}_0 \arrow[r] & \widetilde{LC}_{-1} \arrow[r] & 0
\end{tikzcd}$$

$T_{-1} = 0$ and for $\lambda : \Delta_n \to Y$, it is possible to define;

$$T_\lambda := b_\lambda(\lambda - T\partial\lambda)$$

We now verify that $\partial T + T\partial = \mathbb{1} - S$ on $L\tilde{C}_n Y$ by induction on n. For $\lambda : \Delta_n \to Y$;

$$\partial T(\lambda) = \partial(b_\lambda(\lambda - T\partial\lambda)) = \lambda - T\partial\lambda - b_\lambda \partial(\lambda - T\partial\lambda)$$

Using that $\partial b + b\partial = 1$ and that $\partial\lambda \in \widetilde{LC}_{n-2} Y$. Then;

$$\begin{aligned}
&= \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \partial\lambda - \partial T(\partial\lambda)) \\
&= \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \partial\lambda + S(\partial\lambda) + T(\partial\partial\lambda) \\
&= \lambda - T\partial\lambda - S(\lambda)
\end{aligned}$$

Using that $S(\lambda) = b_\lambda(S(\partial\lambda))$. Then it follows that;

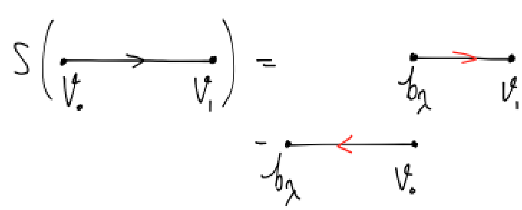
$$\partial T\lambda + T\partial\lambda = \lambda - S(\lambda)$$

$\square$

Example 35. *Let $\lambda = [v_0, v_1] : \Delta_1 \to Y$. Then;*

$$\begin{aligned}
S_\lambda = b_\lambda S(\partial\lambda) = b_\lambda S([v_1] - [v_0]) & \\
&= b_\lambda([v_1] - [v_0]) \\
&= [b_\lambda v_1] - [b_\lambda v_0]
\end{aligned}$$

This can be seen graphically as;



Barycentric Subdivision on $C_n(X)$

We now work with LC_n not $\widetilde{LC}_nY$, and we have that $S : LC_n \to LC_nY$ and $T : LC_nY \to LC_{n+1}1$. And, $\partial T + T\partial = \mathbb{1} - S$. We define the operatores;

$$S : C_nX \to C_nX; \quad S_\sigma = \sigma_\# S\Delta_n$$

And,

$$T : C_nX \to C_{n+1}X; \quad T_{\sigma_\#} T\Delta_n$$

Where we make the following notes on the notation. $F : Z \to W$ implies that $f_\# : C_nZ \to C_nW$, where $f_\# = f \circ \sigma : \Delta^n \to X$. We can then think of

$$\Delta^n \in LC_n\Delta_n \subset C_n\Delta_n \overset{\sigma_*}{\to} C_nX$$

. And also that

$$S\Delta_n \in LC_n\Delta_n \overset{\sigma_\#}{\to}$$

Similarly to the previous cases we have the following key properties.

Lemma 6. *1. $S\partial = \partial S$*

2. $\partial T + T\partial = \mathbb{1} - S$

Where these are on C_nX.

Proof. (1), for $\sigma : \Delta_n \to X$,

$$\begin{aligned}
\partial S\sigma &= \partial\sigma_\# S\Delta_n \\
&= \sigma_\# \partial S\Delta_n \\
&= \sigma_\# S\partial\Delta_n \\
&= \sigma_\# \sum (-1)^i \Delta_n^i \\
&= \sigma_\# \sum (-1)^i S\Delta_n^i \\
&= \sum (-1)^i S(\sigma|\Delta_n^i) \\
&= S \sum (-1)^i \sigma|\Delta_n^i \\
&= S\partial(\sigma)
\end{aligned}$$

(2) We follow a similar argument.

$$\begin{aligned}
\partial T(\sigma) =&\partial\sigma_\# T(\Delta_n) \\
&= \sigma_\# \partial T(\Delta_n) \\
&= \sigma_\#(\Delta_n - S\Delta_n - T\partial\Delta_n) \\
&= \sigma - S\sigma - T\partial\sigma
\end{aligned}$$

□

Iterated Subdivision

We consider the case where a simplex is sudivides more than once. Moreover we show that the following operator is a chain homotopy from $\mathbb{1}$ to S^n.

$$D_n = \sum_{0 \le i \le n} TS^i$$

That is that $\partial D_n + D_n\partial = \mathbb{1} - S^n$.

Proof.

$$\begin{aligned}\partial D_m + D_m \partial &= \sum_{0\le i\le n} (\partial T S^i + T S^i \partial)\\ &= \sum_{0\le i\le n} (\partial T S^i + T \partial S^i \partial)\\ &= \sum_{0\le i\le n} (\partial T + \partial T) S^i\\ &= \sum_{0\le i\le n} (\mathbb{1} - S) S^i = \mathbb{1} - S^n\end{aligned}$$

□

Recall the Lebesque Covering Lemma.
Lemma 7. *Let M be a compact metric space and $V = \{V_j\}$ is an open cover on M. Then there exists ϵ such that for all $B(X, \epsilon)$ in M is contained in some V_j.*

We apply this result to $M = \Delta^n$, where $V = \{\sigma^{-1}(U_i) | U_i \in U\}$ and use that barycentric subdivision reduces the diameter of a simplex.
Then by compactness of Δ_n it follows that;

$$\forall \sigma : \Delta_n \to X, \quad \exists n(\sigma) : S^{n(\sigma)}(\sigma) \in C_n^U X$$

for each σ. Then let $n(\sigma)$ be the smallest such power.

Definition 38. *We define $D : C_n X \to C_{n+1} X$ on σ by $D\sigma = D_{n(\sigma)}\sigma$*

Then we get that;

$$\partial D_{n(\sigma)} + D_{n(\sigma)} = \mathbb{1} - S^{n(\sigma)}$$

where $(\partial D + D\partial)\sigma = \partial D_{n(\sigma)}\sigma + D\partial\sigma = \sigma - [S^{n(\sigma)}\sigma + D_{n(\sigma)}\partial\sigma - D\partial\sigma]$.
We define;

$$\rho : C_n X \to C_n X \quad \text{to be } \rho(\sigma) = S^{n(\sigma)}\sigma + D_{n(\sigma)}\partial\sigma - D\partial\sigma$$

And we claim that;

$$(D_n(S) - D)(\partial\sigma) \in C_n^u X$$

To prove this recall that;

$$\partial\sigma = \sum_i (-1)^i \sigma_i; \quad m(\sigma_i) \le m(\sigma)$$

Then from this it follows that;

$$D_{n(\sigma)}\sigma_j = \sum_{0\le j\le m(\sigma)} T S^i$$

And,

$$D\sigma_j = \sum_{0\le i\le m(\sigma_j)}^{TS_i}$$

Then together these imply that;

$$(D_{m(\sigma)} - D)\sigma_j = \sum_{m(\sigma_j)} \le i \le m(\sigma) T S^i \sigma_j \in C_n^U(X)$$

for some n as required.

Next we claim that ρ is a chain map. This is proven using the result that $\partial D + D\partial = \mathbb{1} - \rho$. Then;

$$\partial\rho\sigma = \partial\sigma - \partial D\partial\sigma - D\partial^2\sigma = \rho\partial\sigma$$

So now we have established the chain map $\rho : C_n X \to C_n^U X$ for all n. It then follows immediately that

$$\rho \circ \sigma = \mathrm{id}_{C_n^U X} \quad \text{and} \quad \partial D + D\partial = \mathbb{1} - i\rho$$

We now move on to stating and proving the Excision Theorem.

4.4.2 The Excision Theorem

Theorem 18 (Excision theorem)**.** *Suppose $Z \subset A \subset X$, and $\bar{Z} \in \mathring{A}$ then, for all n;*

$$H_n(X, A) = H_n(X \backslash Z, A \backslash Z)$$

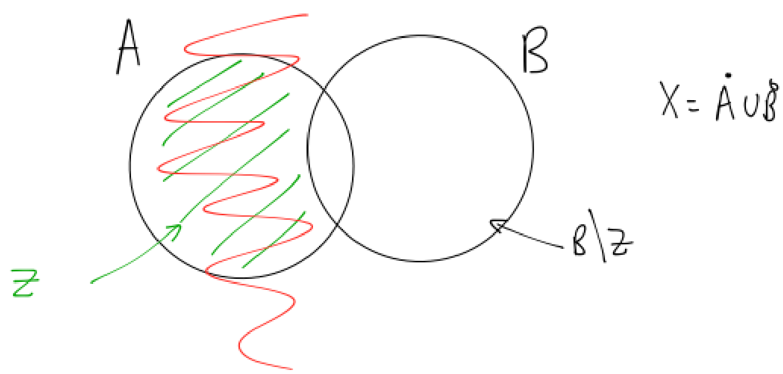


Figure 4.9: A graphical visualisation of the Excision Theorem

Proof. We aim to apply the results above relating to Barycentric subdivision to the case when $U = \{A, B\}$. We will write $C_n(A + B) = C_n^U X$ then we get for all n;

$$C_n(A + B) \hookrightarrow C_n X$$

is a homotopy equivalence. And so i induces the following isomorphism between the homology groups for all n;

$$i : C_n(A+B)\big/C_nA \to C_nX/C_nA$$

The natural chain map $C_nB \to C_n(A + B)$ induces a chain map for all n;

$$C_nB\big/C_n(A+B) \to C_n(A+B)\big/C_nA$$

Moreover, this is an isomorphism since;

$$C_nB\big/C_n(A+B) = \bigoplus_{\substack{\sigma:\Delta\to B \\ \sigma(\Delta^n)\not\subseteq A}} \mathbb{Z}[\sigma]$$

And;

$$C_n(A+B)\big/C_n(A) = \bigoplus_{\substack{\sigma:\Delta\to B \\ \sigma(\Delta^n)\not\subseteq A}} \mathbb{Z}[\sigma]$$

And the map sends a basis 1 to 1 to a basis and so;

$$H_n(B, A \cap B) \xrightarrow{\simeq} H_n(X, A)$$

□

4.5 The Equivalence of Simplicial and Singular Homology

Lemma 8 (5-lemma). *Suppose given a commutative diagram (of abelian group) with exact rows:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\alpha"] & C \arrow[r, "k"] \arrow[d, "\alpha"] & D \arrow[r, "l"] \arrow[d, "\alpha"] & E \arrow[d, "\alpha"] \\ A' \arrow[r, "i"] & B' \arrow[r, "j"] & C' \arrow[r, "k"] & D' \arrow[r, "l"] & E' \end{tikzcd}$$

$\alpha, \beta, \delta, \epsilon$ *isomorphism* $\implies \gamma$ *isomorphism*

1. β*,* δ *onto,* ϵ *injective* $\implies \gamma$ *onto*
2. β, γ *injective,* α *onto* $\implies \gamma$ *injective*

Proof. follow your none diagram chase. Let's prove 2:
Suppose $\gamma(x) = 0$
$0 = k'\gamma(x) = \delta k(x)$ δ injective $\implies k(x) = 0 \implies x = j(y)$
$0 = \gamma(x) = \gamma \circ j(y) = j'\beta(y)$ so $\beta(y) \in ker(j') = Im(i')$ so $\exists z \in A'$ such that $i(z) = \beta(y)$
α onto $\implies z = \alpha(\omega)$ from some $\omega \in A$
$\beta i(\omega) = i'\alpha(\omega) = \beta(y) \implies i\omega - y \in ker(\beta)$
β injective $\implies i\omega = y$ but $x = ju = ji\omega = 0$ so $x = 0$ i.e γ is injective. □

Theorem 19. $(X, A)\Delta$*-complex point. The natural chain map* $C_n^\Delta(X, A) \to C_n(X, A)$ *induces isomorphism* $H_n^\Delta(X, A) \to H_n(X, A)$ *(for* $A = \emptyset$ *we get* $H_n^\Delta X \simeq H_n$*)*

Proof. Look at long exact sequence of the pair X_k, X_{k-1}

$$\begin{tikzcd} H_n^\Delta(X_k, X_{k-1}) \arrow[r, "\delta"] \arrow[d] & H_n^\Delta(X_{k-1}) \arrow[r] \arrow[d, "\simeq"] & H_n^\Delta(X_k) \arrow[r] \arrow[d] & H_n^\Delta(X_k, X_{k-1}) \arrow[r, "\delta"] \arrow[d] & H_n^\Delta(X_{k-1}) \arrow[d, "\simeq"] \\ H_n(X_k, X_{k-1}) \arrow[r, "\delta"] & H_n(X_{k-1}) \arrow[r] & H_n(X_k) \arrow[r] & H_n(X_k, X_{k-1}) \arrow[r, "\delta"] & H_n(X_{k-1}) \end{tikzcd}$$

By 5-lemma $H_n^\Delta X_k \xrightarrow{\simeq} H_n X_k$ In general, for a $\Delta-$complex, the continuos map:

$$\left(\bigsqcup \Delta_k(\alpha), \bigsqcup \partial\Delta_k(\alpha)\right) \xrightarrow{\sqcup\alpha} (X_k, X_{k+1})$$

induces:

$$\bigsqcup \Delta_k(\alpha) \Big/ \bigsqcup \partial\Delta_k(\alpha) \simeq X_k \Big/ X_{k-1}$$

$$\begin{aligned}\implies &= H_n({X_k}\big/{X_{k-1}}) = H_n({X_k}\big/{X_{k-1}}) = H_n({\sqcup \Delta_k(\alpha)}\big/{\sqcup \partial \Delta_k(\alpha)})\\ &= \bigoplus H_n(\Delta_k(\alpha), \partial\Delta_k(\alpha))\\ &= \begin{cases} \bigoplus \mathbb{Z}[\Delta_k(\alpha)] & \text{if } n = k\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

We have:

$$C_n^\Delta(X_k, X_{k-1}) = \begin{cases} \bigoplus \mathbb{Z}e_n(\alpha)] & \text{if } n = k\\ 0 & \text{otherwise}\end{cases}$$

To finish:

1. Suppose X is finite dimensional then $X = X_k$ for some k & by above $H_n^\Delta = H^\Delta X_k = H_n X_k = H_n X$
2. X infinite dimensional

$\square$

4.6 Degree of a map

Definition 39. *For a map $f : S^n \to S^n (n \geq 1)$, the induced map $f_* H_n(S^n) to H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself and so must be of the form $f_*(x) = dx$ for some integer d depending only on f. This integer is called the degree of f.*

Proposition 9. *Here some properties of d:*

1. $deg\mathbb{1} = 1$
2. $deg f = 0$ *if f is not surjective. If we choose a point $x \in S^n \backslash f(S^n)$ the f can be factored as a composition*

   $$S^n \to S^n\backslash\{x\} \hookrightarrow S^n$$

   $H_n(S^n\backslash\{x\}) = 0$ *since the space $S^n\backslash\{x\}$ is contractible. Therefore we have:*

   $$H_n(S^n) \to (0) \to H_n(S^n)$$

   Therefore $f_* = 0$
3. *If* $f \simeq g \implies deg(f) = deg(g)$
4. $deg(f \circ g) = deg(f) \cdot deg(g)$
5. *Let $S^n = \{v | \|v\| = 1\} \subset \mathbb{R}^{n+1}$*
   Let $\tau : S^n \to S^n$ be the restricting of $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_{n-1}, -x_n)$ then $deg(\tau) = -1$.
   This is a Δ-complex structure on S^n with two dimensional cells $S^n = \Delta_n(1) \cup \Delta_n(2)$

   $$C_\cdot^\Delta : 0 \to \mathbb{Z}[\Delta_n(1)] \oplus \mathbb{Z}[\Delta_n(2)] \xrightarrow{\partial} \bigoplus_{i=0}^{n} \mathbb{Z}[\Delta_n^i(1)] \to \bigoplus \mathbb{Z} \to \dots$$

   $H_n(S^n) = H_n^\Delta(S^n = ker(\partial) = \mathbb{Z}[\Delta(1) - \Delta(2)]$
   $\tau(\Delta_n(1) - \Delta_n(2)) = [\Delta_n(2) - \Delta_n(1)] = [\Delta_n(1) - \Delta_n(2)]$
6. *Let $a : S^n \to S^n$ be the antipodal map then $deg(a) = (-1)^{n+1}$ $(a = \tau_0\tau_1\tau_2 \dots \tau_n \implies deg(a) = (-1)^{n+1})$*

7. f *has no fixed points* $\implies$ $deg(f) = (-1)^{n+1}$
   If f *has no fixed point then* $f \sim a$ *(the antipodal map). The segment (in* $\mathbb{R}^{n+1}$*)* $[f(x)_* = -x]$ *does not contain* $\bar{0} \in \mathbb{R}^{n+1}$

$$[0,1] \ni t \to \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|} \in S^n$$

   is a homotpy from f*.*

4.7 Cellular Homology

We recall that a CW complex is built inductively as $\cup X_k$ (the k-skeleton of X). Where;

$$X_k = X_k \underset{\sqcup\phi_\alpha}{\sqcup} D_k(\alpha)$$

and $\phi_\alpha : \partial D_k(\alpha) = S^{n-1}(\alpha) \to X_{k-1}$. The following lemma is useful in calculating the homology groups.

Lemma 9. *The pair* (X_k, X_{k-1}) *is a good pair.*

Proof. Consider the following 'proof by picture' in figure **??** $\square$

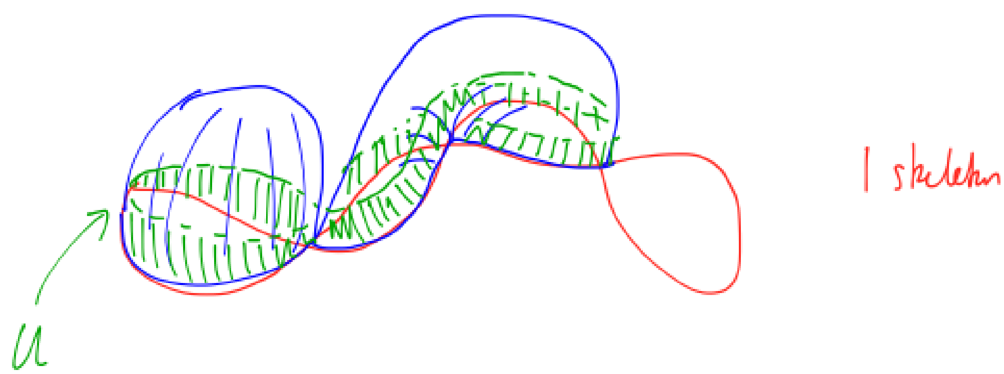


Figure 4.10: X_k, X_{k-1} is a good pair

Lemma 10.

1. $H_n(X_k, X_{k-1}) = \begin{cases} (0) & \text{if } n \neq k \\ \bigoplus \mathbb{Z}[D_n(\alpha)] & \text{if } n = k \end{cases}$
2. $H_n(X_k) = (0)$ *for* $n > k$
3. $i : X_k \subset X$ *induces* $i_* : H_n(X_k) = H_n(X)$ *for* $n < k$

Proof.

1. $H_n(X_k, X_{k-1}) = \widetilde{H_n}(X_k/X_{k-1}) = \widetilde{H_n}(\vee S^n(\alpha))$
2. $H_{n+1}(X_k, X_{k-1}) \to H_n(X_{k-1}) \to H_n(X_k) \to H_n(X_k, X_{k-1})$
   - If $n \neq k, k-1$ then by 1 the outer 2 groups are (0).
   - If $n > k$: $H_n(X_{k-1}) = \cdots = H_n = (0)$

3. If $n < k$, $H_n = H_n(X_{k+1}) = \cdots = H_n(X)$ if X is finite dimensional. If X is infinite dimensional don't worry about it. (Just work as for Δ-complexes, work with C_0X a chain with compact support in X and so it meets finitely many cells therefore it is contained in X_k for some finite k)

$$
\begin{tikzcd}
0 \arrow[dr] & & H_n(X_{n+1}) = H_nX \arrow[r] & 0 & \\
& H_n(X_n) \arrow[ur] & & & \\
\cdots \arrow[r] & H_{n+1}(X_{n+1}, X_n) \arrow[u] \arrow[r] & H_n(X_n, X_{n-1}) \arrow[r] \arrow[d, "\delta_n"] & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r] & \cdots \\
& & H_{n-1}(X_{n-1}) \arrow[ur, "i_{n-1}"] \arrow[dr] & & \\
& 0 \arrow[ur] & & H_{n+1}(X_n) = H_n(X) \arrow[d] & \\
& & & 0 &
\end{tikzcd}
$$

Then we have that; $C_n^{cw} = H_n(X_n, X_{n-1})$, $d_n = i_{n-1} \circ \delta_n$

□

Remark 4. *It is a complex and* $H_n(C_\cdot^{cw} \cdot X) = H_nX$

Proof. Exercise: a little diagram chase □

There are advantages to this theory, namely $C_\cdot^{cw}$ can be considered to be very small but we however need to compute d_n. This is only a superficial disadvantage since there are methods of computing such a boundary map.

So to overcome this disadvantage we need an efficient way of computing d_n:
When $n = 1 H_1(X_1, X_0) \to H_0(X_0)$ is pretty easy.
And when $n \geq 1$, we use the result of the following proposition known as the 'cellular boundary formula'. This makes use of the concept of 'degree' of a map introduced earlier.

Proposition 10. *The* ***Cellular Boundary Formula****:*

$$d_n(D_n(\alpha))) = \sum_\beta d_{\alpha\beta} D_{n-1}(\beta)$$

where

$$d_{\alpha\beta} = deg\Big(S^n(\alpha) = \partial D_n(\alpha) \xrightarrow{\phi_\alpha} X_{n-1} \to {X_{n-1}}\big/{X_{n-1} \backslash D_{n-1}(\beta)} = S^{n-1}(\beta)\Big)$$

This is often just expressed as;

$$d_{\alpha\beta} = deg(\Delta_{\alpha\beta})$$

Where

$$\Delta_{\alpha\beta} S^n(\alpha) = \partial D_n(\alpha) \xrightarrow{\phi_\alpha} X_{n-1} \to {X_{n-1}}\big/{X_{n-1} \backslash D_{n-1}(\beta)} = S^{n-1}(\beta)$$

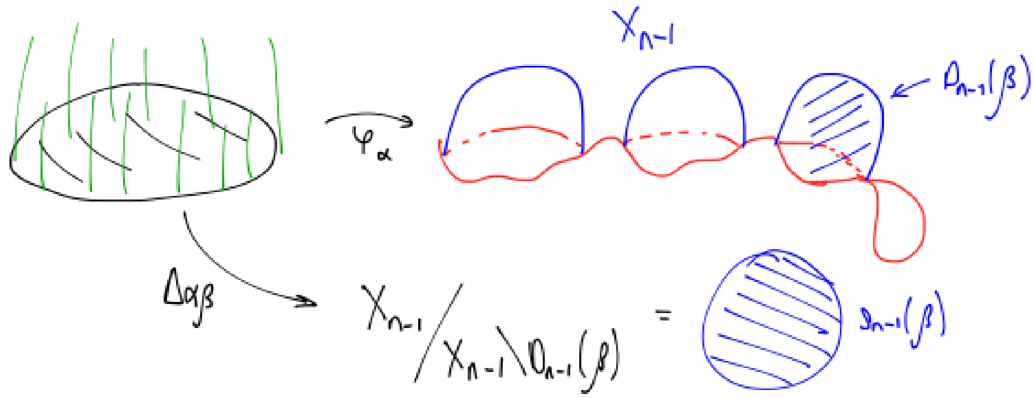


Figure 4.11: A proof of the cellular boundary formula

Proof. A graphical view of what is happening in the proof can be seen in figure 4.11.

Denote by $\phi_\alpha : (D_n(\alpha), \partial D_n(\alpha)) \to (X_n, X_{n-1})$ the obvious continuous map. Also denote by $X_{n-1} \xrightarrow{q} X_{n-1}\big/ X_{n-2} \xrightarrow{q_\beta} S^{n-1}(\beta)$, the obvious collapsing map.

Then the proof of the cellular boundary formula comes from a diagram chase on the following commutative diagram, where we assume that $n > 1$.

$$\begin{tikzcd}
\mathbb{Z} \simeq H_n(D^n(\alpha), \partial D^n(\alpha)) \arrow[r, "\simeq"] \arrow[d, "\phi_{\alpha *}"] & H_{n-1}(\partial D^n(\alpha)) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\phi_{\alpha *}"] & H_{n-1}(S^{n-1}(\beta)) \\
\oplus\mathbb{Z}[D_n(\alpha)] \simeq H_n(X_n, X_{n-1}) \arrow[r, "\delta_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "q_*"] \arrow[d, "\Delta_{n-1 *}"] & H_{n-1}(X_{n-1}\big/ X_{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\simeq"] \\
 & H_{n-1}(X_{n-1}, X_{n-2})) \arrow[r, "\simeq"] & H_{n-1}\left(X_{n-1}\big/ X_{n-2}, X_{n-2}\big/ X_{n-1}\right)
\end{tikzcd}$$

Where $d_n(\alpha, \beta) = d_{\alpha\beta}$, follows from the commutativity of the diagram. $\square$

We now consider techniques for calculations with Cellular Homology.

Proposition 11. *Given $f : S^n \to S^n$ suppose $\exists y \in S^*$ such that $f^{-1} = \{x_1, \dots, x_n\}$ is finite. Let $V \ni y$ be a small open disk and $U_i \ni x_i$ small disks $f(U_i) \subset V \implies f : U_i, U_i\ \{x_i\} \to V, V\ \{y\}$ induces $f_* : H_*(U_i, U_i\ \{x_i\}) \to (V, V\ \{y\})$ therefore $f_* X = d_i X$ and $d_i = deg_{x_i} f$ = local degree of f at x_i*

Proof. Stare long enough at the following commutative diagram:

$$\begin{tikzcd}
 & H_n(U_i, U_i\ \{x_i\}) \arrow[r, "d_i"] \arrow[dl, "\simeq"'] \arrow[d, "k_i"] & H_n(V, V\ \{y\}) \arrow[d, "\simeq"] \\
H_n(S^n, S^n)\backslash\{x_i\}) & H_n(S^n, S^n\backslash f^{-1}(x)) \arrow[l, "p_i"] \arrow[r, "f_*"] & H_n(S^n, S^n\backslash\{y\}) \\
 & H_n(S^n) \arrow[u, "j"] \arrow[ul, "\simeq"] \arrow[r, "d"] & H_n(S_n) \arrow[u, "\simeq"']
\end{tikzcd}$$

By excision,

$$H_n(S^n, S^n\ f^{-1}(y)) = \bigoplus H_n(U_i, U_i\ \{x_i\})$$

Therefore we have:

1. By excision $H_n(S^n, S^n\backslash f^{-1}(y)) = \bigoplus\limits_{i=0,\ldots,n} H_n(U_i, U_i\backslash\{x_i\}) = \bigoplus_{i=1}^{n} \mathbb{Z}$
2. By $\mathbb{Z}$ of upper square $f_* k_i(1) = d_i$
3. $J(1) = \overbrace{(1, 1, \ldots, 1)}^{\text{n times}} = \sum_{i=1}^{n} k(1)$ indeed $p_i j(1) = 1$
4. By commutativity of the lower square $d = f_* J(1) = f_*(\sum k_i(1)) = \sum_{i=1}^{n} d_i$

□

Remark 5. *Let $x \in S^n$ and $x \in U \subset S^n$ a small disk. Then:*

$$H_n S^n \simeq H_n(S^n, S^n\ \{x\}) = H_n(U, U\ \{x\})$$

by excision

Example 36.

1. *there are $f : S^n \to S^n$ of any given degree of $\mathbb{Z}$. Let $A = S^n$ ($\sqcup$ of k disjoint disks)*

$$S^n \to S^n/A = V_k S^n \to V_k S^n \to S^n$$

Therefore by proposition above $deff = (k-h) - h$

2. *$f : S^1 \ni z \mapsto z^d \in S^1$ degree d.*
3. *$f : \mathbb{C} \to \mathbb{C}$ a polynomial of degree d implies $\tilde{f} : S^2 \to S^2$ where $S^2 = \mathbb{C} \cup \{\infty\}$ such that $\tilde{f}(\infty) = \infty$ and $deg\tilde{f} = d$*

We now consider the computation of the cellular homology groups for a few standard examples.

Example 37. Firstly, consider the standard orientated surface of genus g, $\Sigma_g =$
Then as we have done before we can impose a CW complex on this surface such that there are 1 0-cell, $2g$ 1-cells and 1 2-cell.

We then get that, $C_2 = \mathbb{Z}, C_1 = \mathbb{Z}^{2g}, C_0 = \mathbb{Z}$, which form the following exact sequence;

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z}$$

Where by the Cellular Boundary Formula $d_2 = 0$ and so;

$$H_0(\Sigma_g) = \mathbb{Z}, H_1(\Sigma_g) = \mathbb{Z}^{2g}, H_2(\Sigma_g) = \mathbb{Z}$$

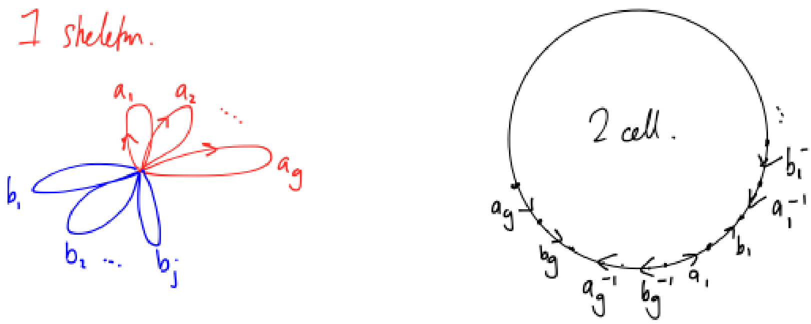


Figure 4.12: A CW structure on Σ_g

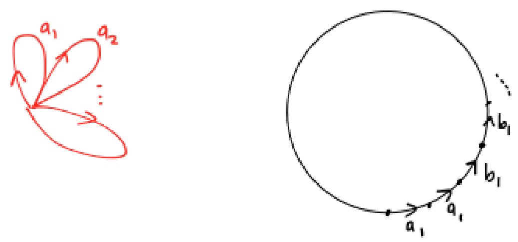


Figure 4.13: A CW complex on a non-orientable surface of genus g

Example 38. We now consider the case of a non-orientated surface of genus g. The cellular decomposition in figure 4.13, gives us 1 0-cell, g 1-cells and 1 2-cell. Therefore we have that;

$$C_2 = \mathbb{Z} = C_0, C_1 = \mathbb{Z}^g$$

Which fits into the following sequence,

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z}^g \xrightarrow{\cdot 0} \mathbb{Z}$$

Where $f : 1 \mapsto (2, 2, \ldots, 2)$.
Therefore it follows that;

$$H_2 \simeq (0), H_1 \simeq {}^{\mathfrak{g}-\mathbb{1}} \oplus {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}, H_0 \simeq \mathbb{Z}$$

4.7.1 Cellular Homology of $\mathbb{P}^n(\mathbb{R})$

Throughout this argument we will be using two equivalent definitions of the projective space at the same time. These are

$$\mathbb{P}^n(\mathbb{R}) = {}^{S^n}\!/_{x \sim -x} = D_n \underset{\phi}{\cup} \mathbb{P}^{n-1}(\mathbb{R})$$

where $\phi : \partial D_n = S^{n-1} \to \mathbb{P}^{n-1}(\mathbb{R})$. This gives us the CW structure and inductively we get that $\phi_{n-1} : S^{n-1} \to \mathbb{P}^{n-1}(\mathbb{R}) = X_{n-1}$, which is the $n-1$ skeleton of $X = \mathbb{P}^n(\mathbb{R})$. In order to establish the cellular homology groups, we need to calculate the boundary maps in C_n^{CW}. Recall from the cellular boundary formula that;

$$d_{n+1} = deg(S^n \to \mathbb{P}^n(\mathbb{R}) \to {}^{\mathbb{P}^n(\mathbb{R})}\!\Big/_{X_{n-1}} \simeq S^n$$

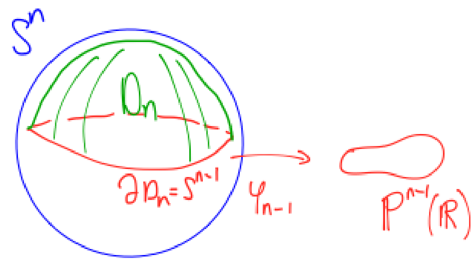


Figure 4.14: A CW structure of the projective real plane

Call the map from $S^n \to S^n$ given above f_n. Then by the degree formula it is sufficient to consider the the local degrees at N, S, where $f_n^{-1}(N) = \{N, S\}$.

To compute the local degree at N, we consider;

$$H_n(S^n) \simeq H_n(D_n^+, D_n^+ \backslash \{N\}) \xrightarrow{\mathrm{id}_*} H_n(D_n/\partial D_n, D_n/\partial D_N \backslash N) = H_n(S^n)$$

And this implies that $deg_N(f_n) = 1$.

Similarly we compute the local degree at S, where we observe the above sequence is obtained by first composing with the antipodal map which has degree $(-1)^{n+1}$. Therefore $deg_S(f_n) = (-1)^{n+1}$. This can be seen in figure 4.15.

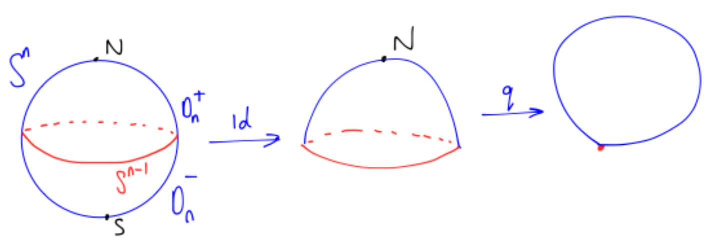


Figure 4.15: A picture of what is going on

Where on $D_n^+ \subset S^n$, $f_n = q \circ \mathrm{id}$ where $q : D_n \to D_n/\partial D_n \simeq S^n$. And on $D_n^- \subset S^n, F_n = q \circ \mathrm{id} \circ a$ where $a : D_n^- \to D_n^+$.

Therefore;

$$d_{n+1} = deg_N(f_n) + deg_S(f_n) = 1 + (-1)^{n+1}$$

Hence the boundary map is either the zero map or multiplication by 2. This we get the following cellular chain complex for $\mathbb{P}^n(\mathbb{R})$.

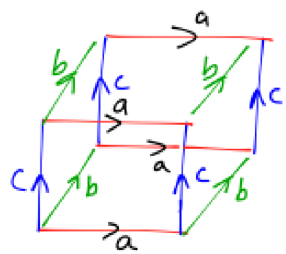
$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Hence it follows that;

$$H_k\mathbb{P}^n(\mathbb{R}) = \begin{cases} \mathbb{Z} & \text{for } k = 0, k = n \text{odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

Cellular Homology of the 3d torus

The 3d torus is characterised by $S^1 \times S^1 \times S^2$. And so this has the CW complex of; Hence the



following sequence is produced;

$$\mathbb{Z} \overset{d_3=0}{\to} \mathbb{Z}^3 \overset{d_2=0}{\to} \mathbb{Z}^3 \overset{d_1=0}{\to} \mathbb{Z}$$

Where the boundary maps $d_1 = 0$ and $d_2 = 0$ as for the torus. But we claim that $d_3 = 0$, we attach the two skeleton by imagining the cube above on the surface of a sphere. And since the local degrees are either +1 or -1, it follows that $d_3 = 0$.

Therefore the homology groups are as follows;

$$H_0 = \mathbb{Z}, H_1 = \mathbb{Z}^3, H_2 = \mathbb{Z}^3, H_3 = \mathbb{Z}^3$$

4.8 Mayer-Vietoris Sequences

We state and prove the existence of Mayer-Vietoris sequences. These can be thought of as "Van-Kampen for homology groups".

Theorem 20. *Let $A, B \subset X$ such that $X = \mathring{A} \cup \mathring{B}$ then there exists an exact sequence of;*

$$H_n(A \cap B) \overset{\phi}{\to} H_n(A) \oplus H_n(B) \overset{\psi}{\to} H_n(X) \overset{\delta}{\to} H_{n-1}(A \cap B) \to \dots$$

Where $\phi(x) = (i_{1*}x, -i_{2*})$ and $\psi(y, z) = j_{1*}y + j_{2*}z$

Then;

Proof. We have that;

$$C_n(A + B) = \bigoplus_{\sigma:\Delta_n \to X, \sigma(\Delta_n) \subset A \cup B} \mathbb{Z}[\sigma]$$

Recall that we already know that $C_n(A + B) \subset C_n(X)$ is a homotopy equivalence of complexes. (We used this in excision). And it is clear that we have an exact sequence of complexes;

$$(0) \to C_n(A \cap B) \overset{\alpha}{\to} C_n(A) \oplus C_n() \overset{\beta}{\to} C_b(A + B) \to (0)$$

Where $\alpha(x) = (x, -x)$ and $\beta(y, z) = y + z$.

This gives us the Mayer-Vietoris Sequence. □

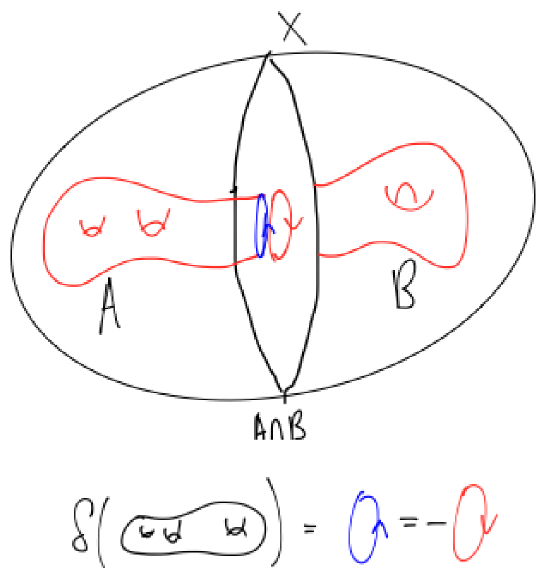
It is often useful to be able to construct the map $\delta : H_nX \to H_{n-1}(A \cap B)$. For this, suppose $z = x + y \in Z_n(A + B)$ with $x \in C_n(A)$ and $y \in C_n(B)$. That is;

$$\partial z = \partial x + \partial y = 0 \in C_{n-1}(A + B)$$

and

$$\partial x = -\partial y \in C_{n-1}(A \cap B)$$

Then it follows that $\delta[z] = [\partial x] = [-\partial y]$. We now consider an example of using the Mayer-Vietoris



Sequence.
Example 39. *Let K be the klein bottle which can be characterised as $K = A \cup B$ where A and B are both Möbius strips joined at their boundaries with a suitable overlap. Then A, B and $A \cap B$ are homotopically equivalent to circles, so $H_1(A) = H_1(A \cap B) = H_1(B) \simeq \mathbb{Z}$. Hence the sequence becomes;*

$$0 \to H_2(K) \to H_1(A \cap B) \xrightarrow{\phi} H_1(A) \oplus H_2(B) \to H_1(K) \to 0$$

Now consider the map $\phi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, is a 2-to-1 map, it therefore has degree 2. And since this map is injective it follows that $H_2(K) = 0$ and $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$. Moreover all higher homology groups of K are zero from the earlier part of the Mayer-Vietrois sequence.

Exercise: Trying calculating the homology groups this way for a surface of genus 2.

4.9 The Euler Characteristic

For a CW comples the atextbfEuler Characteristic, $\chi(X)$ is defined to be the sum $\sum_n (-1)^n c_n$ where c_n is the number of n cells of X. However this result can be generalised for homology groups as follows. IF H is a finitely generated abelian group then $H \simeq \mathbb{Z} \oplus T$ where T is a finite group and $r = rank(H)$. Then we can define the Euler characteristic as;
Definition 40. *Let $C_.$ be a complex of abelian groups with finitely generate homology. Then;*

$$e = \sum_{i=0}^{\infty} r_k H_i$$

Note that $r_i H_i$ is also known as the ith Betti number
Lemma 11. *Suppose that $C_i = (0)$ for $i \in [0, n]$ and all C_i are finitely generated then;*

$$e = \sum (-1)^i r_k C_i$$

Proof. This was done in a homework assignment for vector spaces however this more general case is not harder. ☐

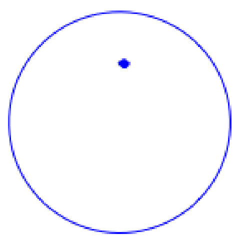
This lemma becomes useful when X is a CW complex of finite dimension with finitely many cells in each dimension since;

$$C_i^{CW} = H_i(X_i, X_{i-1}) = \bigoplus_{\alpha} \mathbb{Z}[D_i(\alpha)]$$

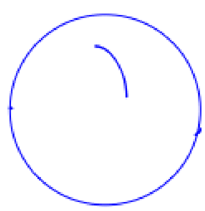
And so;

$$e(X) = \sum_{i=0}^{\infty} (-1)^i \#\{i - cells\} = \sum_{i=0}^{\infty} (-1)^i r_k H_i X$$

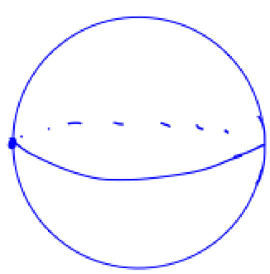
This is a homotopy invariant.
Example 40. *Consider the CW structures on S^2. Then;*



$e = 1 + 0 + 1 = 2$



$e = 2 - 1 + 1 = 2$



$e = 1 - 1 + 2 = 2$

4.10 Comparison between π_1 and H_1.

Proposition 12. *There is a (unique) abelian group G^{ab} (the abelianization of G) & homomorphism $G \to G^{ab}$ characterised by the universal property:*
$\forall$ abelian group A & homomorphism $f : G \to A$ there $\exists!$ homomorphism $g : G^{ab} \to A$ such that the following diagram commutes:

Proof. In two steps: construction of G^{ab}, prove that the construction satisfies the universal property.

1. let $[G, G] \trianglelefteq G$ be the normal subgroup of G generated by all commutators $[a, b] = aba^{-1}b^{-1} \in G$ of 2 elements $a, b \in G$:
$$G^{ab} = G \big/ [G, G]$$
2. $f : G \to A$ abelian $\implies f([G, G]) = \{e\} \implies [G, G] < N = ker(f)$
$\implies f : G \to A$ by elementary algebra.

□

Theorem 21. *X path connected, $x_0 \in X \implies \pi_1(X, x_0)^{ab} = H_1X$*

Proof. We define a group homomorphism $\phi : \pi_1(X, x_0) to H_1X$. We consider maps $f : \Delta_1^{[v_0,v_1]} \to X$ and we think of them in two ways: either a path $f : I \to X$ from $f(v_0)$ to $f(v_1)$ (and sometimes a loop) or a element of C_1X a 1-dimensional chain in X. We establish the following:

1. f constant $\implies f \sim 0$
2. $f \simeq g \implies f \sim g$
3. for composable paths $f \cdot g \sim f + g$
4. $\bar{f} \sim -f$

we now prove the 4 facts above:

1. $f(\Delta_1) = x \in X$
$C_1X \ni f = f_\#(x)$ where $x \in C_1\{pt\}$ & $f_\# : C.\{pt\} \to C.X$ is a chain map.
2. $f \simeq g$ this means $\exists$ homotopy $\Delta_1 \times I \to X$
Goal: to manufacture a singular 2-chain $y \in C_2X$ such that $\partial Y = f - g$
we do that as in the picture.
$\sigma_1 = [v_0v_1w_1]$ $\sigma_2 = [v_0w_0w_1]$
$Y = [F : \sigma_1 \to X] - [F : \sigma_2 \to X]$
$\partial Y = [F : [v_1, w_1] \to X] - [F : [v_0w_1] \to X] + [F : [v_0v_1] \to X] - [F : [w_0w_1] \to X]$
3. Construct $F : \Delta_2 \to X$ as $F = f \cdot g \circ p$
where $f \cdot [v_0v_2] \to X$ is the product of the two paths & p:$\Delta_2 \to [v_0v_2]$ is the orthogonal projection of Δ_2 to $[v_0v_2]$ (mapping in particular v_1 to the midpoint of $[v_1v_2]$) $[F : \Delta_2 \to X] \in C_2X$ $\partial F = g - f \cdot g + f$ ie $f \cdot g \sim f + g$
4. Apply 3 to $g = \bar{f}$
$f\bar{f} \sim f + \bar{f}$
also by 2 $f\bar{f} \sim e \implies f \cdot \bar{f} \sim 0 \implies 0 \sim f\bar{f} \sim f + \bar{f}$ ie 4

We do get indeed a group homomorphism $h : \pi_1(X, x_0) \to H_1X$
We claim h is surjective:
take $\gamma = \sum n_i\sigma_i \in Z_1X$ $\sigma_1 : \Delta_1 \to X$ we may assume $n = \pm 1$ using 4. I may assume all $n_i = +1$ if σ_i appears in γ with coefficient $n_i = -1$ I use $-\sigma_i \sim +\bar{\sigma}_i$. Using 3 I can make daisy chain of σ_1 and assume that all σ_i are loops (not necessarily at x_0. σ_i starts at $p_i \in X$ and ends at $p_{i+1} \in X$

$\partial\sigma = \sum \partial\sigma_i = \sum(p_i - p_{i+1}) = 0$
we may assume there is only one loop $\gamma = \sigma$ (not necessarily based at x_0)
by 3 and 4 $\alpha \cdot \sigma \cdot \bar{\alpha} \sim \alpha + \sigma - \alpha = \sigma$ therefore h is surjective.
Suppose $fL\Delta_1 \to X$ is a loop at x_0 (in part $f \in Z_1X$) then if $h(f) = 0$ in H_1 then $f \in [-\pi, \pi]$. I can write σ as a word w in π_1 where if γ appears somewhere in w, then γ^{-1} also appears somewhere else in w. There exists a 2-chain $\sigma = \sum n_i\sigma_i$ with $\partial\sigma = f$ by deconstructing σ if necessary may assume all n_i are equal to $n_i = \pm 1$. In fact we may assume all $n_i = \pm 1$

$$\partial\sigma = \sigma[v_1v_2] - \sigma[v_0v_2] + \sigma[v_0v_1]$$

We have there $\sigma_i : \Delta_2(i) \to X$. Assemble all these simplices $\Delta_2(i)$ int oa 2d Δ-complex K

we know

$$f = \partial(\sum \sigma_i) = \sigma_{i,j}(-1)^j\tau_j(i)$$

group all but one of $\tau_j(i)$ into pairs where the one that's left has $\tau_j(i) = f$

I use all my $\Delta_2(i)$ to asssemble a Δ-complex K with $\partial K = f$
If $\sigma(K_0) = x_0$ then we are done
$\gamma abcb^{-1}a^{-1}c^{-1} \simeq \sigma \implies \gamma = cabc^{-1}b^{-1}a^{-1}\ w$ in $\pi_1(X, x_0)$
We want to construct a homotpy $F : K \times I \to X$ rel γ $(\gamma \subset K)$
$F(y, 0) = \sigma$
$F(y, 1) = \sigma^{-1}$ and $\sigma(k_0) = x_0$
It is easy to contruct F on k_0. We want to extend the homotopy. $\square$

Definition 41. *This is the homotopy extension property of a pair Y, A , $A \subset Y$ closed.*
$f_0 : Y \to X$
$F : A \times I \to X$